Section 2.1

Introduction to Derivatives

Now that we understand limits, it is time to return to our main goal: we want to study rates of change of functions. In section 1.4, we wanted to know the instantaneous rate of change of the function $f(x) = 100\sqrt{x}$ at $x = 1$. We learned that instantaneous rate of change of $f(x)$ at $x = a$ is the same as the slope of a tangent line to $f(x)$ at the point $(a, f(a))$. In the example above, we saw that we should compute the slope of the tangent line shown below in order to find the instantaneous rate of change of $f$:

![Graph showing tangent line](image)

Slope of the tangent line at $x = 1$ is the instantaneous rate of change of $f$ at $x = 1$.

So answering the question comes down to finding the slope of the line above.

Since we could not actually calculate the slope of the tangent line above, we used secant lines to help us out. For example, we calculated the slope of the tangent line below, and used its slope as an approximation for the slope of the tangent line:

![Graph showing secant line](image)

Finally, we saw that we could increase the accuracy of our approximation for the slope of the tangent line by sliding the second point of the secant closer to the point of tangency:
The idea of sliding this second point closer to \((1, f(1))\) is exactly the idea behind a limit.

**Example.** Use limits of slopes of secant lines to find the *exact* value for the slope of the tangent line to \(f(x) = 100\sqrt{x^2}\) at \(x = 1\).

Let’s start by writing a general equation for the slope of a secant line. On the graph below we have drawn a secant line whose slope we will compute:

Recall that we want to be able to move the second point of the secant closer to \((1, f(1))\); so instead of picking a specific second point, we will give ourselves a bit of “wiggle room” by choosing the \(x\) coordinate of the second point to be \(1 + h\). The height of the curve at \(x = 1 + h\) is

\[
f(1 + h) = 100\sqrt{\frac{1 + h}{2}}.
\]

Let’s fill in this information on the graphic:
We want to compute the slope of the secant line passing through the two points above. Since
\[
\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1},
\]
the slope is
\[
\text{slope} = \frac{100\sqrt{\frac{1+h}{2}} - 100\sqrt{\frac{1}{2}}}{1 + h - 1}.
\]
Let’s simplify this quantity.
\[
\frac{100\sqrt{\frac{1+h}{2}} - 100\sqrt{\frac{1}{2}}}{1 + h - 1} = \frac{100}{\sqrt{2}} \frac{(\sqrt{1+h} - 1)}{h} = \frac{100(\sqrt{1+h} - 1)}{h\sqrt{2}}.
\]
Now think about what happens when we make $h$ smaller:
The second point on the secant line slides back towards the first point, and the secant line itself looks more and more like the tangent line. In other words, the slope of tangent to $f(x)$ at $x = 1$ is exactly

$$\lim_{h \to 0} \frac{100(\sqrt{1+h} - 1)}{h\sqrt{2}}.$$ 

So in order to finish off this problem we need to calculate this limit.

Let’s analyze the function whose limit we’re evaluating,

$$\frac{100(\sqrt{1+h} - 1)}{h\sqrt{2}}.$$ 

We should notice immediately that 0 is not in the domain of the fraction above; in fact, $h = 0$ causes both the numerator and the denominator to be 0. So we have no idea what the limit as $h$ approaches 0 is (we don’t even know if the limit exists!), and in order to be able to say anything at all about the limit, we must rewrite the function.

Since there is a radical in the numerator of the fraction, we should utilize the trick of multiplying
So the slope of the tangent line is $\frac{100}{2\sqrt{2}}$.

In Section 1.4, we used slopes of secants to predict that the slope of the tangent line would be about 35.36. Now we have computed the exact value for the slope of the tangent line—it is extremely close to our earlier guess!

Above, we calculated the limit

$$\lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}$$

in order to find the instantaneous rate of change of $f(x)$ at $x = 1$. This limit is important enough that we will give it a name, the derivative, and new notation in the definition below:

**Definition 4.** The derivative of a function $f(x)$ at $x = a$, denoted by $f'(a)$, is the number

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

if the limit exists.
Alternatively, we can write
\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.
\]

The derivative of \( f(x) \) at \( x = a \) is the number that tells us

- The instantaneous rate of change of \( f(x) \) at \( x = a \)
- The slope of the line tangent to \( f(x) \) at \( x = a \).

In the example above, we found that \( f'(1) = \frac{100}{2\sqrt{2}} \).

**Example.** Find the derivative of \( f(x) = x^{-3} \) at \( x = -1 \), and interpret your answer.

We need to use the limit definition above,
\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},
\]
with \( f(x) = x^{-3} \) and \( a = -1 \). Notice that it will be easier to think of \( f(x) \) as \( f(x) = 1/x^3 \). Let’s fill in the limit with our data:

\[
f'(-1) = \lim_{h \to 0} \frac{f(-1 + h) - f(-1)}{h}
= \lim_{h \to 0} \frac{\frac{1}{(h-1)^3} - \frac{1}{(-1)^3}}{h}
= \lim_{h \to 0} \frac{\frac{1}{(h-1)^3} + 1}{h}
= \lim_{h \to 0} \frac{\frac{1}{(h-1)^3} + \frac{(h-1)^3}{(h-1)^3}}{h}
= \lim_{h \to 0} \frac{\frac{1 + (h-1)^3}{(h-1)^3}}{h}
= \lim_{h \to 0} \frac{\frac{1 + (h - 1)^3}{h(h - 1)^3}}{h}
= \lim_{h \to 0} \frac{1 + (h^3 - 3h^2 + 3h - 1)}{h(h - 1)^3}
= \lim_{h \to 0} \frac{1 + h^3 - 3h^2 + 3h - 1}{h(h - 1)^3}
= \lim_{h \to 0} \frac{h^3 - 3h^2 + 3h}{h(h - 1)^3}
= \lim_{h \to 0} \frac{h^2 - 3h + 3}{h(h - 1)^3}
= \lim_{h \to 0} \frac{0^2 - 3 \cdot 0 + 3}{(0 - 1)^3}
= \frac{3}{-1}
= -3.
\]
So our conclusion is that $f'(1) = 3$; so the slope of the line tangent to $f(x) = x^{-3}$ at $x = -1$ is $-3$. The curve $f(x) = \frac{1}{x^3}$ and its tangent line at $x = -1$ are graphed below:

Based on this graph, our answer of $f'(-1) = -3$ seems reasonable. The tangent line above does indeed appear to have a slope of $-3$.

**Example.** Suppose that the revenue from selling a DSLR cameras is

$$R(x) = 200 \left(1 - \frac{1}{x}\right),$$

where $x$ is the number of cameras sold. Determine the rate at which revenue is changing when the 20th camera is sold.

The problem asks us to find the rate of change of revenue when $x = 20$. We are simply being asked to determine $R'(20)!$
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The derivative formula tells us that this is

\[ R'(20) = \lim_{h \to 0} \frac{R(20 + h) - R(20)}{h} \]

\[ = \lim_{h \to 0} \frac{200(1 - \frac{1}{20+h}) - 200(1 - \frac{1}{20})}{h} \]

\[ = \lim_{h \to 0} \frac{200(\frac{20}{20+h} - \frac{1}{20+h}) - 200(\frac{20}{20} - \frac{1}{20})}{h} \]

\[ = \lim_{h \to 0} \frac{200(\frac{19+h}{20+h}) - 200(\frac{19}{20})}{h} \]

\[ = \lim_{h \to 0} \frac{200(19+h)}{20+h} - \frac{200 \cdot 19}{20} \]

\[ = \lim_{h \to 0} \frac{200(19+h)}{20+h} - 190 \]

\[ = \lim_{h \to 0} \frac{200(19+h) - 190(20+h)}{20+h} \]

\[ = \lim_{h \to 0} \frac{200(19 + h) - 190(20 + h)}{h(20 + h)} \]

\[ = \lim_{h \to 0} \frac{3800 + 200h - 3800 - 190h}{h(20 + h)} \]

\[ = \lim_{h \to 0} \frac{200h - 190h}{h(20 + h)} \]

\[ = \lim_{h \to 0} \frac{10h}{h(20 + h)} \]

\[ = \lim_{h \to 0} \frac{10}{20 + h} \]

\[ = \frac{10}{20 + 0} \]

\[ = \frac{1}{2} \]

We see that \( R'(20) = .5 \), which means that selling the 20th camera only changes the revenue by 50 cents.