Section 1.8

**Continuity**

In the previous section, we saw that we could determine limits for many different functions just by evaluating the function. For example, we looked at the function $f(x) = x^4 + 3x - 2$ and saw that

$$\lim_{x \to 1} f(x) = f(1) = 2.$$ 

Our interpretation of this phenomenon is that the function behaves exactly the way that we expect the function to behave. This is an extremely important concept, and we give this idea a name:

**Definition 1.** A function $f(x)$ is continuous at the number $x = a$ if the following conditions are satisfied:

1. $a$ is in the domain of $f(x)$
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$. 

In other words, a function is continuous at a number $x = a$ in its domain if it is "smooth" near $a$; there are no holes or gaps, no unexpected behavior. I should be able to trace the graph of $f(x)$ through the number $x = a$ without having to lift my pencil.

If $f(x)$ is defined near $x = a$ but is not continuous at $x = a$, then we say that $f$ is discontinuous at $x = a$.

**Example.** The function $f(x)$ is graphed below. Determine all points at which $f(x)$ has a discontinuity, and explain why each point is a discontinuity.

In order to draw the curve $f(x)$, there are four different numbers at which you would have to lift up your pencil and move it to another location:

a. $x = -4$
At each of these numbers, the function has a gap; it behaves unexpectedly. Indeed, the function $f(x)$ does have a discontinuity at each of the numbers above. Let’s inspect each one closely to determine why.

a. $x = -4$: It appears that $x = -4$ is not in the domain of $f(x)$; so $f$ is not continuous at $x = -4$.

b. $x = -1$: While the function is defined at $x = -4$, it does not behave as we expect it to. In other words, $\lim_{x \to -1} f(x) \neq f(-1)$.

c. $x = -2$: Again, $f(x)$ is defined at $x = 2$, and again the function does not behave well. In particular, $\lim_{x \to 2} f(x)$ does not exist, so $f$ has a discontinuity at $x = -2$.

d. $x = 5$: The function is not defined here, thus $f(x)$ has a discontinuity.

Think about the graph of $f(x)$ once more—we have enumerated all of the points at which $f$ has a discontinuity; in other words, $f$ is continuous at each number $x \neq -4, -1, 2, 5$.

![Graph of f(x)](image)

$f$ is continuous at all numbers except $-4, -1, 2, 5$.

There is a nicer way to say this, using the definition below:

**Definition 3.** A function $f$ is *continuous on an interval $(a, b)$* if it is continuous at each number in the interval.
Your book also discusses continuity at endpoints of intervals, but we will not consider this idea here.

**Example.** The graph of the function $f(x)$ is given below. Determine all intervals on which $f(x)$ is continuous.

We have already seen that the only discontinuities of $f(x)$ occur at $x = -4, -1, 2, \text{ and } 5$, so $f(x)$ is continuous on the intervals

$$(-\infty, -4), (-4, -1), (-1, 2), (2, 5), \text{ and } (5, \infty).$$

**Example.** For each function listed below, determine a number at which the function is not continuous, and explain why the numbers cause discontinuities.

1. $f(x) = \frac{1}{x}$
2. $g(x) = \begin{cases} x + 1 & \text{if } x \geq 1 \\ -x - 1 & \text{if } x < 1. \end{cases}$
3. $h(x) = \begin{cases} \frac{x^2 - 1}{x+1} & \text{if } x \neq -1 \\ 1 & \text{if } x = -1. \end{cases}$

1. The first example is easy— we know that $f(x)$ is *not defined* when $x = 0$, so $f$ has a discontinuity at $x = 0$. 

2. The function \( g(x) \) is defined for all real numbers \( x \), but as with other piecewise-defined functions we have studied so far, we know that the number 1 at which the rule for \( g(x) \) changes could cause the function to act strangely. Let’s try evaluating left-hand and right-hand limits at \( x = 1 \). On the left of \( x = 1 \), the function behaves like \(-x - 1\), and so

\[
\lim_{{x \to 1^-}} g(x) = \lim_{{x \to 1^-}} (-x - 1) = -2.
\]

On the right of \( x = 1 \), the function behaves like \( x + 1 \), so

\[
\lim_{{x \to 1^+}} g(x) = \lim_{{x \to 1^+}} (x + 1) = 2.
\]

The left-hand and right-hand limits do not match up at \( x = 1 \), so the limit does not exist.

Since the limit at \( x = 1 \) does not exist, \( f(x) \) is not continuous at \( x = 1 \).

3. Notice that \( h(x) \) is defined for all real numbers; however, it is again a piecewise defined function, and the number \( x = -1 \) where the rule for \( h \) changes may cause difficulties. We should attack the problem by evaluating left-hand and right-hand limits at \( x = -1 \). On the left-hand side of \(-1\), \( h \) behaves like \( \frac{x^2 - 1}{x + 1} \); so

\[
\lim_{{x \to -1^-}} h(x) = \lim_{{x \to -1^-}} \frac{x^2 - 1}{x + 1}
\]

Unfortunately, \( x = -1 \) is not in the domain of the fraction—it causes the denominator to be 0. However, \( x = -1 \) also causes the numerator to be 0, so the limit may or may not exist. We can try factoring and rewriting to find a simpler function with the same limit. Since

\[
\frac{x^2 - 1}{x + 1} = \frac{(x - 1)(x + 1)}{x + 1} = x - 1 \text{ whenever } x \neq -1,
\]

we know that

\[
\lim_{{x \to -1^-}} \frac{x^2 - 1}{x + 1} = \lim_{{x \to -1^-}} (x - 1) = -2.
\]

Let’s consider the right-hand limit now; since \( h \) behaves like \( \frac{x^2 - 1}{x + 1} \) on the right-hand side of \( x = -1 \), we know that

\[
\lim_{{x \to -1^+}} h(x) = \lim_{{x \to -1^+}} \frac{x^2 - 1}{x + 1}.
\]

Now we could go through all of the calculations for the limit, but it should be clear that this limit will work exactly like the previous one—other words,

\[
\lim_{{x \to -1^-}} \frac{x^2 - 1}{x + 1} = \lim_{{x \to -1^+}} \frac{x^2 - 1}{x + 1} = -2,
\]

which means that the general limit exists:

\[
\lim_{{x \to -1}} \frac{x^2 - 1}{x + 1} = -2.
\]

However, let’s compare this with the actual value of \( h(x) \) at \( x = -1 \): the rule tells us that \( h(-1) = 1 \). So since

\[
\lim_{{x \to -1}} h(x) \neq h(-1),
\]

\( h(x) \) has a discontinuity at \( x = -1 \).
We are now ready to significantly simplify our study of limits. From the material that we have seen so far in this section, we know that, if the function $f(x)$ is defined and continuous at the number $x = a$, then the limit of $f(x)$ as $x \to a$ matches up with $f(a)$. In other words, if we know that a function is continuous at $x = a$, then *taking the limit as $x \to a$ is as simple as evaluating $f(a)$*. When this idea is combined with the following theorem, we will have the power to quickly evaluate a wide variety of limits.

**Theorem 7.** The following types of functions are continuous at every point in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

So if $f(x)$ is any of the above type of function, and $x = a$ is in its domain, we know that

$$\lim_{x \to a} f(x) = f(a),$$

so that we do not actually have to evaluate a *limit* at $x = a$—we merely need to evaluate the function at $x = a$. In fact, sums, differences, products, quotients (with non-zero denominators) and constant multiples of any of the functions above behave the same way.

**Example.** Evaluate the following limits, if possible:

1. $\lim_{x \to 2} \frac{1}{\sqrt{x^2 - 1}}$

2. $\lim_{x \to 1} \frac{1}{\sqrt{x^2 - 1}}$

3. $\lim_{x \to -2} x^3 - \sqrt{2x} + \frac{3x}{2}$

4. $\lim_{x \to \pi} \cos \theta$

5. $\lim_{x \to 0} \sec \theta$

6. $\lim_{x \to \pi/2} \tan \theta$

1. $\lim_{x \to 2} \frac{1}{\sqrt{x^2 - 1}}$: Since $\frac{1}{\sqrt{x^2 - 1}}$ is continuous and $x = 2$ is in its domain,

$$\lim_{x \to 2} \frac{1}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{(2)^2 - 1}} = \frac{1}{\sqrt{3}}.$$
2. \( \lim_{x \to 1} \frac{1}{\sqrt{x^2 - 1}} \): Again, \( \frac{1}{\sqrt{x^2 - 1}} \) is continuous, but \( x = 1 \) is not in its domain, so we cannot use the theorem above to evaluate the limit; instead, we must go back to the information we learned in the previous section. Since \( x = 1 \) causes the denominator of the fraction to be 0, we should check the numerator. Here, the numerator is the constant 1, so we know that 
\[
\lim_{x \to 1} \frac{1}{\sqrt{x^2 - 1}} = \text{DNE.}
\]
Again, we might be able to say more about why the limit does not exist; let’s check left-hand and right-hand limits at \( x = 1 \). To check the left-hand limit, we will plug in numbers close to 1 but slightly smaller, say .8, .9, etc. In this case, \( \frac{1}{\sqrt{x^2 - 1}} < 0 \) so that 
\[
\lim_{x \to 1^-} \frac{1}{\sqrt{x^2 - 1}} = -\infty.
\]
Let’s check the right-hand limit: by plugging in numbers slightly larger than 1 into the fraction, we see that \( \frac{1}{\sqrt{x^2 - 1}} > 0 \), so that 
\[
\lim_{x \to 1^+} \frac{1}{\sqrt{x^2 - 1}} = \infty.
\]
Since the left-hand and right-hand limits do not match up, we will leave our answer as 
\[
\lim_{x \to 1} \frac{1}{\sqrt{x^2 - 1}} = \text{DNE.}
\]
3. \( \lim_{x \to -2} x^3 - \sqrt{-2x} + \frac{3x}{2} \): The function is a sum of continuous functions, and \( x = -2 \) is in the domain of each one, so
\[
\lim_{x \to -2} x^3 - \sqrt{-2x} + \frac{3x}{2} = (-2)^3 - \sqrt{-2(-2)} + \frac{3(-2)}{2} = -8 - 2 - 3 = -13.
\]
4. \( \lim_{\theta \to \pi} \cos \theta \): The cosine function is continuous, and \( x = \pi \) is in its domain, so
\[
\lim_{\theta \to \pi} \cos \theta = \cos \pi = -1.
\]
5. \( \lim_{\theta \to 0} \sec \theta \): The secant function is continuous, and \( x = 0 \) is in its domain, so
\[
\lim_{\theta \to 0} \sec \theta = \sec 0 = 1.
\]
6. \( \lim_{\theta \to \pi/2^-} \tan \theta \): While the tangent function is continuous, \( \frac{\pi}{2} \) is not in its domain. To evaluate the limit, let’s rewrite the function as
\[
\tan \theta = \frac{\sin \theta}{\cos \theta}.
\]
Then the limit may be rewritten as
\[ \lim_{\theta \to \pi/2^-} \tan \theta = \lim_{\theta \to \pi/2^-} \frac{\sin \theta}{\cos \theta}. \]

Since \( \cos(\pi/2) = 0 \) but \( \sin(\pi/2) \neq 0 \), we know that the
\[ \lim_{\theta \to \pi/2^-} \frac{\sin \theta}{\cos \theta} \text{ DNE}. \]

We can determine why the limit does not exist here by evaluating the function at numbers very close to but just smaller than \( \pi/2 \); the sine and cosine functions are both positive in the first quadrant, and the numerator of the fraction is approaching 1 while the denominator is approaching 0, so
\[ \lim_{\theta \to \pi/2^-} \frac{\sin \theta}{\cos \theta} = \infty; \]
thus
\[ \lim_{\theta \to \pi/2^-} \tan \theta = \infty. \]

Another limit law that will become extremely useful in the future is below—you will need this theorem when you study L'Hôpital’s Rule in Calculus II. The law describes the way in which limits interact with compositions of functions.

**Theorem 8.** Suppose that \( \lim_{x \to a} g(x) = b \), and that \( f(x) \) is continuous at \( b \). Then
\[ \lim_{x \to a} f(g(x)) = f(b); \]
alternatively, we may write
\[ \lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right). \]

Theorem 8 simply says that (in most cases), evaluating the limit of a composition function is as simple as evaluating the limit of the inside function, and plugging the result into the outside function.

**Example.** Evaluate
\[ \lim_{x \to 1} \sin(x^2 - 1). \]

Notice that the function whose limit we want to evaluate is *neither* a trig function nor a polynomial; instead, it is a composition of the two types of functions. We must use Theorem 8 in order to evaluate its limit.

The theorem says that, as long as the sine function is continuous at \( \lim_{x \to 1}(x^2 - 1) \), then
\[ \lim_{x \to 1} \sin(x^2 - 1) = \sin \left( \lim_{x \to 1}(x^2 - 1) \right). \]

So we need to calculate the original limit in two parts:
1. Find the number \( a = \lim_{x \to 1} (x^2 - 1) \)

2. Evaluate \( \sin a \), if \( \sin x \) is continuous at \( x = a \).

1. The first step is easy: \( \lim_{x \to 1} (x^2 - 1) = (1^2 - 1) = 0 \).

2. 0 is in the domain of the sine function (as are all real numbers!), and the sine function is continuous at each point in its domain, so the theorem says that

\[
\lim_{x \to 1} \sin(x^2 - 1) = \sin\left( \lim_{x \to 1} (x^2 - 1) \right) = \sin 0 = 0.
\]

**Intermediate Value Theorem**

To understand the theorem that comes next, let’s think about two contrasting examples.

**Example.** As you drive around Easton in your (ordinary, non-DeLorean) car, your speed is a continuous function of time. Since your speed function is continuous, you’ll notice an interesting phenomenon when you watch your speedometer. As you accelerate, the needle on your speedometer will have to sweep smoothly through its arc from 0 to the speed limit, 30 mph. The speedometer is not allowed to skip any numbers between 0 and 30, and this is due directly to the fact that the speed function is continuous.

**Example.** As Marty McFly drives the DeLorean through the parking lot, he will see a different phenomenon. As he accelerates past 88 mph, he is suddenly able to travel at light speed and into the past. The DeLorean’s speed is not a continuous function of time, which means that the speed function is allowed to skip speeds past 88 and jump straight into light speed.

The first example illustrates the idea presented in the theorem below:

**Intermediate Value Theorem.** Suppose that \( f(x) \) is defined on \([a, b]\) continuous on \((a, b)\). Assume that \( f(a) \neq f(b) \). Then for any number \( N \) between \( f(a) \) and \( f(b) \), there is a number \( c \) between \( a \) and \( b \) so that \( f(c) = N \).

Graphically, the idea of the theorem is that a continuous function must cover all outputs between \( f(a) \) and \( f(b) \) as the input varies from \( a \) to \( b \):

![Diagram showing the Intermediate Value Theorem](image-url)
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The Intermediate Value Theorem applied to your speed function—you had to hit every speed (output) between 0 and 30. However, since the DeLorean’s speed function wasn’t continuous, it was allowed to jump around and skip speeds.

Example. Let \( f(x) = x^4 - 3x^2 + 1 \). Does \( f(x) \) have any real roots? (Hint: evaluate \( f(1) \) and \( f(2) \).)

Keep in mind that a function has a root at the number \( a \) if the function crosses the \( x \)-axis at \( x = a \); in other words, \( f(a) = 0 \). The curve graphed below has a root at \( x = a \):

Our normal method for finding roots of a polynomial is to factor, but that won’t help us here—we don’t know how to factor degree 4 polynomials!

Notice that \( f(1) = 1 - 3 + 1 = -1 \) and \( f(2) = 16 - 12 + 1 = 3 \). Since \( f(x) \) is continuous, the Intermediate Value Theorem says that \( f \) must take on every output between \(-1 \) and \( 3 \)—so in particular there is a number \( c \) between 1 and 2 where \( f(c) = 0 \). This is precisely a root of \( f(x) \). The curve is graphed below: