Advanced Integration Techniques: Partial Fractions

The method of partial fractions can occasionally make it possible to find the integral of a quotient of rational functions. Partial fractions gives us a way to break a fraction into smaller components so that integration is possible. As a simple example, we might rewrite the fraction $\frac{7x-5}{(x+1)(x-2)}$ as

$$\frac{7x-5}{(x+1)(x-2)} = \frac{4}{x+1} + \frac{3}{x-2}.$$

To truly understand partial fraction decomposition, we must remember the algorithm for adding a pair of fractions with different denominators:

In order to add $\frac{4}{x+1} + \frac{3}{x-2}$, the fractions must have a common denominator. The *product* of the two current denominators will work, $(x+1)(x-2)$. To rewrite each fraction with the same denominator, we multiply each by a special form of 1 so that we do not change the fractions, simply make them look different:

$$\frac{4}{x+1} = \frac{4}{x+1} \cdot \frac{x-2}{x-2} = \frac{4x-8}{(x+1)(x-2)}$$

and

$$\frac{3}{x-2} = \frac{3}{x-2} \cdot \frac{x+1}{x+1} = \frac{3x+3}{(x+1)(x-2)}.$$

Now that the fractions have the same denominator, we can add them together:

$$\begin{aligned}
\frac{4}{x+1} + \frac{3}{x-2} &= \frac{4x-8}{(x+1)(x-2)} + \frac{3x+3}{(x+1)(x-2)} \\
&= \frac{4x-8+3x+3}{(x+1)(x-2)} \\
&= \frac{7x-5}{x^2-x-2}.
\end{aligned}$$

When we use partial fractions to decompose a fraction into its components, we will need to reverse the process of adding the component fractions. If we now wished to decompose $\frac{7x-5}{x^2-x-2}$ into its component fractions, we would need to recognize that $\frac{7x-5}{x^2-x-2} = \frac{7x-5}{(x+1)(x-2)}$ was obtained by adding fractions of the form $\frac{A}{x+1}$ and $\frac{B}{x-2}$. We know that

$$\begin{aligned}
\frac{A}{x+1} + \frac{B}{x-2} &= \frac{A(x-2)}{(x+1)(x-2)} + \frac{B(x+1)}{(x+1)(x-2)} \\
&= \frac{Ax-2A}{(x+1)(x-2)} + \frac{Bx+B}{(x+1)(x-2)} \\
&= \frac{Ax-2A+Bx+B}{x^2-x-2} \\
&= \frac{7x-5}{x^2-x-2}.
\end{aligned}$$

The numerators above must be equal, so $Ax - 2A + Bx + B = 7x - 5$. A and B are at present unknown, but we can use the given equation to determine their values. Terms with different powers

of x do not affect each other, so we know that $Ax + Bx = 7x$ and $-2A + B = -5$. We now have a system of equations that needs to be solved:

$$A + B = 7 \qquad \text{(the } x\text{'s appear in each term and can be eliminated)}$$
$$-2A + B = -5.$$

When solving a system of equations, we can always:

1. Multiply a row by a constant, and

2. Add a multiple of one row to another

in order to solve for the variables. In this case, let's eliminate the A and solve for B:

$$
\begin{aligned}
2A + 2B &= 14 \qquad \text{(by multiplying the original first row by 2)} \\
+\underline{-2A + B} &= \underline{-5} \\
3B &= 9 \\
\Rightarrow B &= 3.
\end{aligned}
$$

Now that we know that $B = 3$, it is easy to see that $A = 4$. Thus we have recovered the constants and can rewrite the fraction:

$$\frac{7x - 5}{x^2 - x - 2} = \frac{4}{x + 1} + \frac{3}{x - 2}.$$

When we decompose with partial fractions, we will always start by factoring the denominator as far as possible, into products of linear factors (those of the form $(x - a)$) and irreducible quadratic factors (those of the form $(ax^2 + bx + c)$ which can not be factored into linear factors). We will need to handle certain cases differently, depending on the types of factors that show up in the denominator. The cases are summarized below:

1. If the linear factor $(x - a)$ appears n times, (that is $(x - a)^n$ appears in the factorization of the denominator) then the expanded fraction must contain the corresponding terms

$$\frac{A_1}{(x - a)} + \frac{A_2}{(x - a)^2} + \ldots + \frac{A_n}{(x - a)^n}$$

2. If the irreducible quadratic factor $(ax^2 + bx + c)$ appears n times, (that is $(ax^2 + bx + c)^n$ appears in the factorization of the denominator) then the expanded fraction must contain the corresponding terms

$$\frac{B_1 x + C_1}{(ax^2 + bx + c)} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \ldots + \frac{B_n x + C_n}{(ax^2 + bx + c)^n}.$$

Partial Fraction Decomposition

In order to decompose a fraction, we will need to follow these steps:

1. Reduce the highest power of x in the numerator so that it is **less than** the highest power of x in the denominator by performing long division, if necessary.

2. Factor the denominator as far as possible into products of linear factors and irreducible quadratic factors.

3. Use the process in the previous section to determine the form of the expanded fraction.

4. Add the component fractions.

5. Solve for the unknown values of the numerators using a system of equations.

Examples

Evaluate $\displaystyle\int \frac{x^2 + 5x - 20}{x^3 - 4x^2}dx$.

The easiest way to evaluate the integral will probably be to break up the fraction into its components using partial fractions. The highest power of x in the numerator is already less than the highest power of x in the denominator, so we do not need to worry with division. Let's factor the denominator as far as possible:

$$x^3 - 4x^2 = x^2(x - 4)$$
$$= x \cdot x(x - 4).$$

The linear factor $x\ (= (x-0))$ appears twice, so the corresponding fractions in the decomposition are
$$\frac{A_1}{x} + \frac{A_2}{x^2}.$$

Since the linear factor $(x - 4)$ appears once, the decomposition will have the corresponding factor

$$\frac{A_3}{(x - 4)}.$$

Now we know the form of the decomposition:

$$\frac{x^2 + 5x - 20}{x^3 - 4x^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{(x-4)}$$

$$= \frac{A_1}{x} \cdot \frac{x(x-4)}{x(x-4)} + \frac{A_2}{x^2} \cdot \frac{(x-4)}{(x-4)} + \frac{A_3}{(x-4)} \cdot \frac{x^2}{x^2}$$

$$= \frac{A_1 x^2 - 4A_1 x}{x^3 - 4x^2} + \frac{A_2 x - 4A_2}{x^3 - 4x^2} + \frac{A_3 x^2}{x^3 - 4x^2}$$

$$= \frac{A_1 x^2 - 4A_1 x + A_2 x - 4A_2 + A_3 x^2}{x^3 - 4x^2}$$

$$= \frac{A_1 x^2 + A_3 x^2 - 4A_1 x + A_2 x - 4A_2}{x^3 - 4x^2}.$$

The numerators of the two fractions must be equal, so we see that $x^2 + 5x - 20 = A_1 x^2 + A_3 x^2 - 4A_1 x + A_2 x - 4A_2$, in particular

$$A_1 + A_3 = 1 \qquad \text{(equating terms involving } x^2\text{)}$$
$$-4A_1 + A_2 = 5 \qquad \text{(equating terms involving } x\text{)}$$
$$-4A_2 = -20 \qquad \text{(equating terms involving } x^0\text{)}$$

We see from the last equation in the system that $A_2 = 5$; we may use this to find A_1 in the second equation; $-4A_1 + 5 = 5$ so that $A_1 = 0$. Thus we can determine the value for A_3 using the first equation, $0 + A_3 = 1$ so that $A_3 = 1$.

Thus the decomposition of the original fraction is

$$\frac{x^2 + 5x - 20}{x^3 - 4x^2} = \frac{0}{x} + \frac{5}{x^2} + \frac{1}{(x-4)}$$

$$= \frac{5}{x^2} + \frac{1}{(x-4)}.$$

So

$$\int \frac{x^2 + 5x - 20}{x^3 - 4x^2} dx = \int \frac{5}{x^2} + \frac{1}{(x-4)} dx$$

$$= -\frac{5}{x} + \ln(x-4) + C.$$

Decompose $\dfrac{4x^2 + 3x + 2}{x^3 + x^2 + x} dx$.

Once again, we won't need to divide since the highest power of x in the numerator is 2 and the highest power of x in the denominator is 3. Let's factor the denominator:

$$x^3 + x^2 + x = x(x^2 + x + 1).$$

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We can see that the quadratic factor $(x^2 + x + 1)$ is irreducible (it can not be decomposed into linear factors) by using the quadratic equation:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4}}{2}$$

returns complex roots because of $\sqrt{-3}$.

Thus the form of the fraction we are after is

$$\frac{A}{x} + \frac{Bx + C}{x^2 + x + 1}.$$

Next we need to add the fractions:

$$\begin{aligned}
\frac{A}{x} + \frac{Bx + C}{x^2 + x + 1} &= \frac{A}{x} \cdot \frac{x^2 + x + 1}{x^2 + x + 1} + \frac{Bx + C}{x^2 + x + 1} \cdot \frac{x}{x} \\
&= \frac{Ax^2 + Ax + A}{x^3 + x^2 + x} + \frac{Bx^2 + Cx}{x^3 + x^2 + x} \\
&= \frac{Ax^2 + Ax + A + Bx^2 + Cx}{x^3 + x^2 + x} \\
&= \frac{Ax^2 + Bx^2 + Ax + Cx + A}{x^3 + x^2 + x} \\
&= \frac{4x^2 + 3x + 2}{x^3 + x^2 + x}.
\end{aligned}$$

Since we must have $Ax^2 + Bx^2 + Ax + Cx + A = 4x^2 + 3x + 2$, we can see that

$$\begin{aligned}
A + B &= 4 &&\text{(by equating the } x^2 \text{ terms)} \\
A + C &= 3 &&\text{(by equating the } x^1 \text{ terms)} \\
A &= 2 &&\text{(since } A \text{ is the only term appearing without the variable } x\text{)}
\end{aligned}$$

Now it is easy to see that $C = 1$ and $B = 2$, so the expanded form of the fraction is

$$\frac{4x^2 + 3x + 2}{x^3 + x^2 + x} = \frac{2}{x} + \frac{2x + 1}{x^2 + x + 1}.$$

Thus

$$\begin{aligned}
\int \frac{4x^2 + 3x + 2}{x^3 + x^2 + x} dx &= \int \frac{2}{x} + \frac{2x + 1}{x^2 + x + 1} dx \\
&= \int \frac{2}{x} dx + \int \frac{2x + 1}{x^2 + x + 1} dx \\
&= 2 \ln|x| + \ln|x^2 + x + 1| + C \qquad \text{using the substitution } u = x^2 + x + 1.
\end{aligned}$$

Long Division

As a brief reminder of how to divide polynomials, let's look at the example

$$\frac{2x^4 + 2x^3 - x^2 - 1}{x^2 - 1}.$$

We set it up as

$$x^2 - 1 \overline{) \quad 2x^4 + 2x^3 \quad - x^2 \qquad - 1},$$

leaving a blank space since there is no x^1 in the original numerator.

The x^2 in the divisor can be multiplied by $2x^2$ to match the $2x^4$ in the dividend, so we will multiply $2x^2$ by each portion of the divisor and subtract from the dividend:

$$
\begin{array}{r}
2x^2 \\
x^2 - 1 \overline{) \; 2x^4 + 2x^3 \; - x^2 \qquad - 1} \\
\underline{- 2x^4 \qquad\quad + 2x^2} \\
2x^3 \; + x^2
\end{array}
$$

Now we simply repeat the process until the power of x in the dividend is less than the power of x in the divisor. The x^2 in the divisor can be multiplied by $2x$ to yield the $2x^3$ in the divisor, so again we multiply each portion of the divisor by $2x$ and subtract from the dividend:

$$
\begin{array}{r}
2x^2 + 2x \\
x^2 - 1 \overline{) \; 2x^4 + 2x^3 \; - x^2 \qquad - 1} \\
\underline{- 2x^4 \qquad\quad + 2x^2} \\
2x^3 \; + x^2 \\
\underline{- 2x^3 \qquad\quad + 2x} \\
x^2 + 2x - 1
\end{array}
$$

We repeat the process once more so that the power of x in the dividend becomes less than the power of x in the divisor:

$$
\begin{array}{r}
2x^2 + 2x + 1 \\
x^2 - 1 \overline{) \; 2x^4 + 2x^3 \; - x^2 \qquad - 1} \\
\underline{- 2x^4 \qquad\quad + 2x^2} \\
2x^3 \; + x^2 \\
\underline{- 2x^3 \qquad\quad + 2x} \\
x^2 + 2x - 1 \\
\underline{- x^2 \qquad\quad + 1} \\
2x
\end{array}
$$

So

$$\frac{2x^4 + 2x^3 - x^2 - 1}{x^2 - 1} = 2x^2 + 2x + 1 + \frac{2x}{x^2 - 1}.$$

We can divide $x^3 - 5x$ by $\frac{1}{2}x^2 - 3x$ as follows:

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$$\frac{\phantom{\frac{1}{2}x^2 - 3x)}}{\frac{1}{2}x^2 - 3x \overline{\smash{\big)}\ \begin{array}{r} 2x + 12 \\ x^3 \qquad\quad - 5x \\ \underline{-x^3 + 6x^2} \\ 6x^2 \ \ - 5x \\ \underline{-6x^2 + 36x} \\ 31x \end{array}}}$$

so that

$$(x^3 - 5x) \div (\frac{1}{2}x^2 - 3x) = 2x + 12 + \frac{31x}{\frac{1}{2}x^2 - 3x}.$$