

**Derivatives of Inverse Trig Functions:**

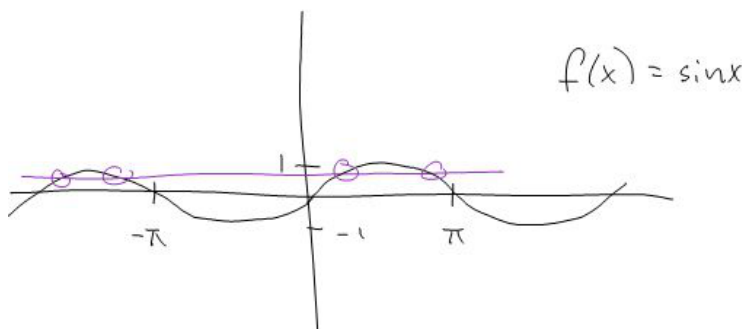
If  $f(x)$  is a function that has an inverse,  $f^{-1}(x)$ , then  $f^{-1}$  "reverses" the action of  $f$ ; if  $f(x) = y$  then  $f^{-1}(y) = x$ . In particular,

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

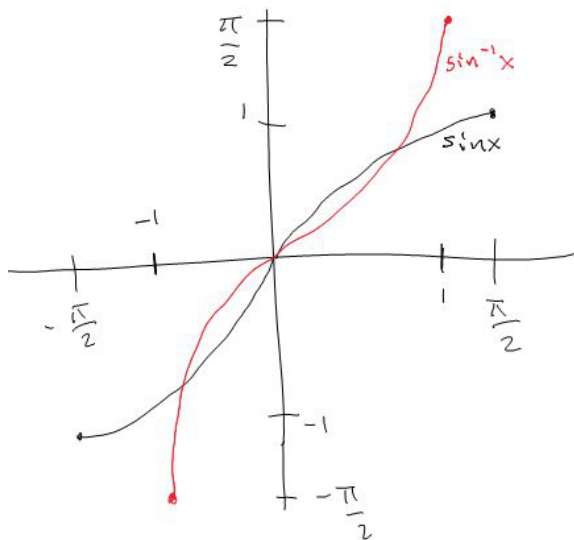
For example,  $f(x) = x^3$  and  $g(x) = \sqrt[3]{x}$  are inverses, since  $f(g(x)) = (\sqrt[3]{x})^3 = x$ ; the two functions "reverse" each other.

In this section we will discuss the inverse trigonometric functions, such as  $\sin^{-1} x$ ,  $\cos^{-1} x$ , etc. In particular, we would like to know the derivatives of these inverse trigonometric functions. Before learning them, however, let's recall a few facts about functions of this type.

None of the trig functions pass the horizontal line test, so technically none of them have inverses.



However, if we restrict the domain of each of the functions, we are able to define what we mean by inverse trig function. For example, by restricting the domain of  $f(x) = \sin x$  to  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ , we get a one-to-one function that does indeed have an inverse:



Recall that the sine function has *angles* as inputs, and outputs a *number* between  $-1$  and  $1$ ; so the inverse sine function accepts *numbers* between  $-1$  and  $1$  as inputs and outputs *angles*. In particular, the domain of the restricted sine function is now the range of  $\sin^{-1} x$ , and the range of  $\sin x$  is the domain of  $\sin^{-1} x$ . Determining the value of  $\sin^{-1} x$  amounts to finding the value for  $y$  in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  so that  $\sin y = x$ ; then  $\sin^{-1} x = y$ .

One minor note: this choice of domain for  $\sin x$  is rather arbitrary; we could choose *any* interval on which  $\sin x$  does not repeat itself as the domain. However, the choice of  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  just happens to be particularly convenient.

We may do the same for each of the trig functions to obtain the inverse trigonometric functions:

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$
$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$

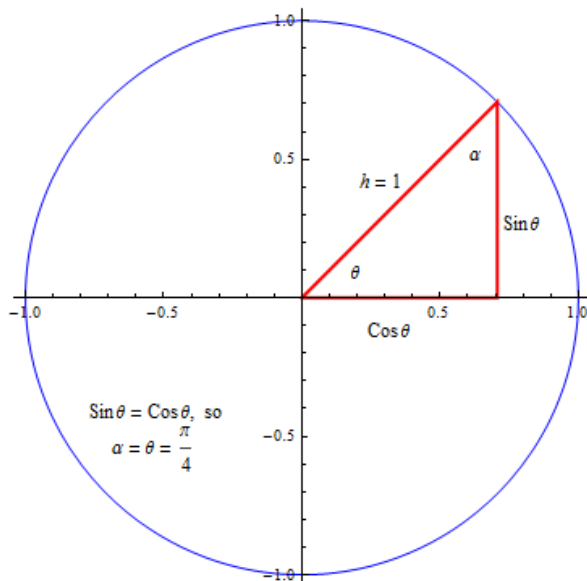
### Examples

Determine  $\tan^{-1}(-1)$ .

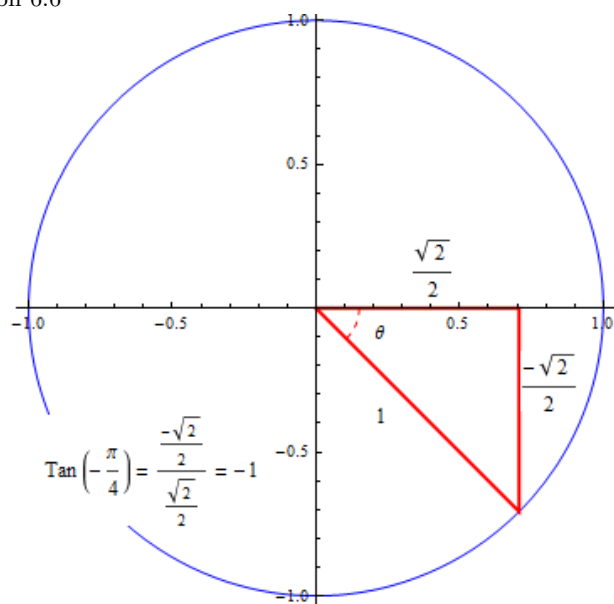
First, let's note that the range of the arctangent function is  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . So finding  $\tan^{-1}(-\frac{\pi}{2})$  is the same as determining a  $\theta$  in  $(-\frac{\pi}{2}, \frac{\pi}{2})$  so that  $\tan \theta = -1$ .

Since the tangent function is negative in the 2nd and 4th quadrants, and the range of the tangent function includes the 1st and 4th quadrants, our answer must be in the fourth quadrant.

For now, let's ignore the negative sign, and simply determine  $y$  so that  $\tan y = 1$ . Since  $\tan \theta = \frac{\sin y}{\cos y}$ , we need to find  $\theta$  so that  $\sin \theta = \cos \theta$ :

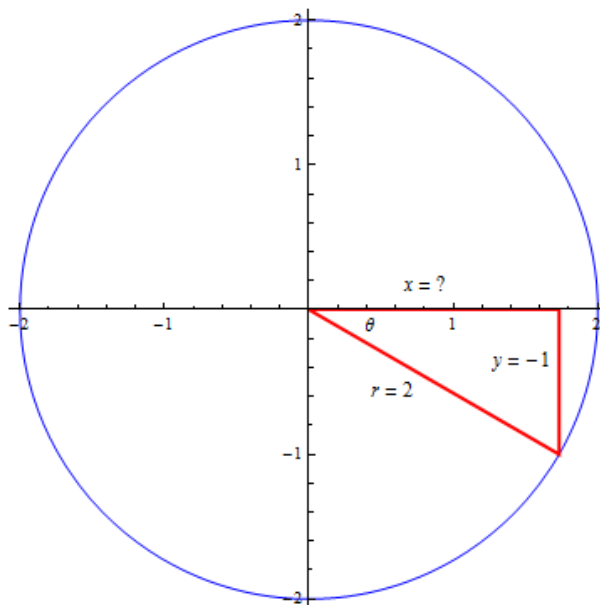


Clearly, this is  $\frac{\pi}{4}$ . Since our answer should be in the 4th quadrant, we change the sign, and get  $\theta = -\frac{\pi}{4}$ .



Find  $\cos(\sin^{-1} \frac{-1}{2})$ .

Since the range of the arcsine function is  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  (the first and fourth quadrants), and the sine function is negative in the third and fourth quadrants, we know that  $\sin^{-1} \frac{-1}{2}$  is in the fourth quadrant. In addition, we know that  $\sin^{-1} \frac{-1}{2}$  is an angle  $\theta$  so that  $\sin \theta = \frac{-1}{2}$ . Recall that  $\sin \theta = \frac{y}{r}$ ; so the following graph can help us to determine the requisite values:



Notice that we don't really have to determine the value for  $\theta$ ; since we merely wish to determine

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$\cos \theta = \frac{x}{r}$ , and we already know  $r = 2$ , we just need to determine  $x$ . Clearly  $x = \sqrt{r^2 - y^2} = \sqrt{4 - 1} = \sqrt{3}$ , so  $\cos(\sin^{-1} \frac{1}{2}) = \frac{\sqrt{3}}{2}$ .

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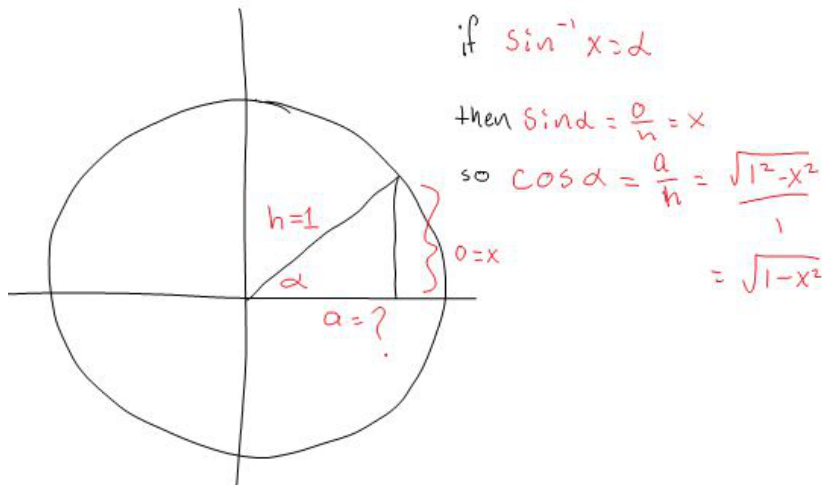
We do not yet know the derivatives of the inverse trig functions, but we can use some basic facts to determine them. For example, consider  $f(x) = \sin x$ , and set  $y = \sin^{-1} x$ . In particular, we know that

$$f(y) = \sin(y) = \sin(\sin^{-1}(x)) = x.$$

Now we're going to be sneaky. Our goal is to **determine**  $\frac{dy}{dx} = \frac{d}{dx} \sin^{-1} x$ ; to do so, let's use implicit differentiation on the equality above,  $\sin y = x$ . Differentiating both sides with respect to  $x$ , we have

$$\begin{aligned} \frac{d}{dx} \sin y &= \frac{d}{dx} x \\ (\cos y) \frac{dy}{dx} &= 1 && \text{(using the chain rule on } \sin y) \\ \frac{dy}{dx} &= \frac{1}{\cos y} && \text{(by dividing both sides by } \cos y) \\ \frac{dy}{dx} &= \frac{1}{\cos(\sin^{-1} x)} && \text{(by rewriting.)} \end{aligned}$$

Now  $\frac{dy}{dx}$  (i.e.  $\frac{d}{dx} \sin^{-1} x$ ) is *precisely* what we want to know, so we're nearly finished; however, we would like to simplify the expression a bit. We can use a unit circle to rewrite  $\cos(\sin^{-1} x)$ :



So  $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$ . Finally, we see that

$$\begin{aligned} \frac{d}{dx} \sin^{-1} x &= \frac{dy}{dx} = \frac{1}{\cos(\sin^{-1} x)} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$


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The remaining derivatives of inverse trig functions may be calculated similarly, and are summarized below:

$$\begin{array}{ll} \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} & \frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2} \\ \frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}} & \frac{d}{dx} \csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}. \end{array}$$


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**Example:** Find  $\frac{d}{dx} \tan^{-1}(3x^2 - x)$ .

We know the derivative of  $\tan^{-1} x$ , but the function we wish to differentiate is more complicated than this; it is actually a composition of the functions  $\tan^{-1} x$  and  $3x^2 - x$ . Thus we will need the chain rule along with the rule for differentiating  $\tan^{-1} x$ . Recall that the chain rule says that

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Using the chain rule, we have

$$\begin{array}{ll} g(x) = 3x^2 - x & g'(x) = 6x - 1 \\ f(x) = \tan^{-1} x & f'(x) = \frac{1}{1+x^2} \\ f'(g(x)) = \frac{1}{1+(3x^2-x)^2}. \end{array}$$

So

$$\frac{d}{dx} \tan^{-1}(3x^2 - x) = \frac{6x - 1}{1 + (3x^2 - x)^2}.$$


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### Integrals related to inverse trig functions

The table of derivatives that we have just learned can help us to determine some integrals that, until now, we would have been unable to compute. In particular, we know that

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

and

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C.$$


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**Examples:** Evaluate  $\int \frac{1}{\sqrt{9-9x^2}} dx$ .

This function does not match any of the functions in our basic list of integrals, but some simple rewriting can fix the problem. Let's factor a 9 out of the radical in the denominator; then

$$\frac{1}{\sqrt{9-9x^2}} = \frac{1}{3\sqrt{1-x^2}}.$$

Now we see that

$$\begin{aligned}\int \frac{1}{\sqrt{9-9x^2}} dx &= \int \frac{1}{3\sqrt{1-x^2}} dx \\ &= \frac{1}{3} \int \frac{1}{\sqrt{1-x^2}} dx \\ &= \frac{1}{3} \sin^{-1} x + C.\end{aligned}$$

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Determine  $\int \frac{3x^2}{1+x^6} dx$ .

On first inspection, this integral looks quite difficult to handle. At this point in the class, we only have two options for integrating it (rewrite so that it looks like a basic integral, or use u-substitution). Substitution seems reasonable; let's try setting  $u = x^3$ , so that  $u^2 = x^6$  and  $du = 3x^2 dx$ . With this substitution, the integral can be rewritten as

$$\begin{aligned}\int \frac{3x^2}{1+x^6} dx &= \int \frac{1}{1+u^2} du \\ &= \tan^{-1} u + C \\ &= \tan^{-1} x^3 + C.\end{aligned}$$