PRIMUS, 23(7): 646–658, 2013
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ISSN: 1051-1970 print / 1935-4053 online
DOI: 10.1080/10511970.2013.764368

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Hands-on SET®

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Abstract: SET® is a fun, fast-paced game that contains a surprising amount of mathematics. We will look in particular at hands-on activities in combinatorics and probability, finite geometry, and linear algebra for students at various levels. We also include a fun extension to the game that illustrates some of the power of thinking mathematically about the game.

Keywords: SET®, combinatorics, geometry, linear algebra, in-class activities.

1. INTRODUCTION

SET® is a fun, fast-paced game. It's a wonderful game for children, who are often able to play better than adults, and it's challenging for all ages. And as we will see, there is a surprising amount of mathematics in the deck of cards.

Each card in the deck has symbols, characterized by four attributes:

- number: one, two, or three symbols;
- color: red, purple, or green;
- shading: empty, striped, or solid;
- shape: ovals, diamonds, or squiggles.

A *set* comprises three cards for which each attribute is independently either all the same or all different. It's important to note that the number of attributes that are the same can vary. In Figure 1, you can see several examples of sets.

To play the game, 12 cards are laid out on a table. Each player looks for sets. The first one who finds a set calls it ("Set!") and takes it. Those three cards

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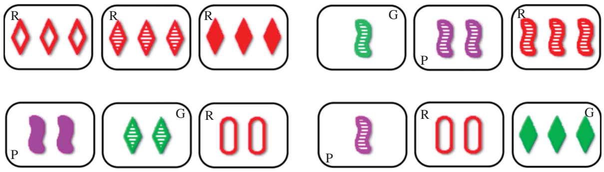


Figure 1. Four examples of sets with differing numbers of common attributes (color figure available online).

are then replaced, and players look for sets in the new layout. If at some point in the game, the players agree that they can't find any sets in the layout, three additional cards are added. If a set is found among this larger group of cards, it is taken and not replaced. The game ends when all players agree that no more sets can be found. The winner is the one with the most sets.

This paper will examine some of the mathematics in the game of SET® (or a sub SET® of the mathematics, if you enjoy terrible puns). We will look in particular at combinatorics and probability, finite geometry, and linear algebra. We finish with a fun extension to the game that illustrates some of the power of thinking mathematically about the game.

2. COMBINATORICS AND PROBABILITY

There are some really lovely ways you can introduce combinatorics and probability to students of any level using the properties of SET®. One of the most important properties (arguably the single most important property) that needs to be introduced before we can begin any kind of discussion is that two cards determine a unique set. We like to call this the Fundamental Theorem of SET®.

The proof is straightforward: given any two cards, if they are the same in an attribute, the third card must also be the same, and if they differ in an attribute, the third card should be the missing expression of that attribute. Thus, the card completing the set is unique. This is analogous to the geometric notion of two points determining a line, and more importantly, it is fundamental to much of the mathematics involved in the game. This idea is important, so it should be introduced to students early, even the first time they are introduced to the game. It's actually a good way to make sure people understand what a set is, as it reinforces the rule that each attribute in the three cards must be all the same or all different.

Once students have played enough to become familiar with the game, you can pose several questions.

1. How many cards are in the deck?
2. How many different sets are there in the deck (allowing overlaps)?
3. How many sets contain a given card?

You can have the students discuss, write out answers, show examples, etc. These basic questions are great to begin with because the answers, and the process of finding the answers, will motivate more in-depth probability questions later.

1. How many cards are in the deck? Use a fundamental counting idea: $3 \times 3 \times 3 \times 3 = 81$.
2. How many different sets are there? We have 81 choices for the first card, 80 choices for the second, and (because of the Fundamental Theorem of SET®) only one for the last card. Because those three cards can be chosen 3! ways (the order of the cards does not matter), the total number of sets is $(81 \times 80)/6 = 1080$.
3. How many sets contain a given card? Once the card is chosen, we have 80 choices for a second card and then only one choice for the card to finish the set. However, since order doesn't matter, we've counted each set twice, so the number of sets containing any given card is $80/2 = 40$.

As students are coming up with answers to these basic questions, they will often start asking their own questions. For example, which type of set is most likely to appear? Would it be a set with all attributes different, or one with two different and two the same, or something else?

To answer this, use standard combinatorial counting techniques to count the number of sets of each type.

All four attributes different:	216 sets	(20% of the deck).
Three attributes different, one the same:	432 sets	(40% of the deck).
Two attributes different, two the same:	324 sets	(30% of the deck).
One attribute different, three the same:	108 sets	(10% of the deck).

3. FINITE GEOMETRY

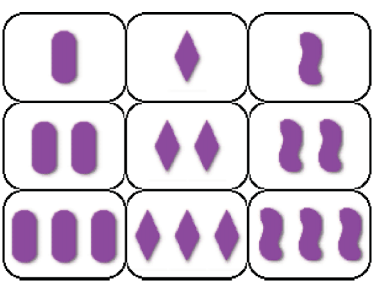
The cards in the game of SET® provide an excellent model for finite affine geometry. The game helps students visualize the geometry, and the geometry provides insight into the game as well. Whether or not your students are at a level to understand what affine geometry is, they will still get a great deal out of this approach. In fact the activity we introduce here was tailored specifically to a group of high school students in a summer enrichment program with great success. We have subsequently done the same activity with a range of students: from fourth and fifth graders in a math club to college students, both first years and students in upper-level courses (we have even done this with college faculty). For students who do know about more abstract geometries, you can have them view the deck as a finite affine geometry: the cards are the points in the geometry, and three points are on a line if those three cards form a set. This means that the deck of cards is really the four-dimensional affine geometry of order 3, $AG(4, 3)$.

Here is the in-class activity to begin familiarizing students with the finite geometry that SET® represents. Ask the students to isolate two attributes, and then find nine cards that have those attributes in common. For example, if they choose to isolate the attributes color and shading, they might do this by choosing the color purple and the solid shading; the nine cards that have two solid purple symbols are shown in Figure 2.

Ask students to organize the cards in a nice way that helps them see all the sets that those nine cards contain. In our experience, the students will almost always make a three-by-three square, and most likely it will look quite similar to the one shown on the left in Figure 3. To the right in the figure, you can see the standard configuration for the affine plane of order 3 (the order tells how many points are on a line). If you look at the figure on the right, you'll see 12 lines – three horizontal lines, three vertical lines, three lines parallel to the main diagonal in bold blue, and three lines parallel to the opposite diagonal. These lines correspond to the 12 sets in the configuration on the left. In fact, this configuration is called a *magic square* on the SET® website [3] (under Teacher's Corner, Academic Research, Mathematical Proof of Magic Squares), because any two cards in the figure make a set with a third card contained in the figure.

This idea can be extended to cards taken from the entire deck, and students enjoy doing this. Take any three cards that are not a set and place them in the upper left corner of a rectangle, as shown in Figure 4 on the left. Find the cards that would complete the horizontal and vertical sets, and then add the card in the center that completes a set with the two new cards. The last three cards are added so every row and column are sets (the order you complete the square doesn't matter, as you can see by inspection). When you are done, you

Figure 2. The nine cards with two solid symbols (color figure available online).



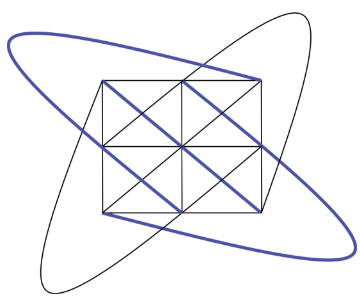


Figure 3. Left: nine cards in a square; right: $AG(2, 3)$ (color figure available online).

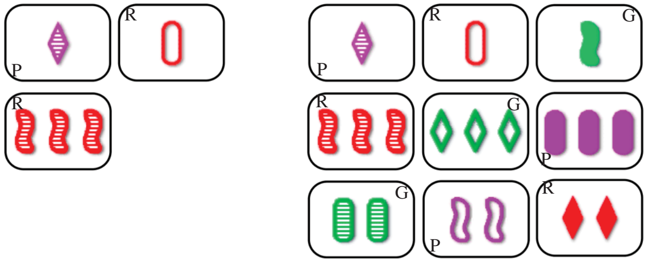


Figure 4. Left: three cards that don't form a set; right: the completed plane of cards (color figure available online).

have a configuration like the one on the right in Figure 4, with sets in all the same places as in Figure 3. Not only do any two cards define a set, any three cards that are not a set define an entire magic square (that is, a plane). The configuration is unique in the sense that if you take any three cards that aren't a set from that square, put them in the upper left and complete a new square as before, it will consist of the same nine cards.

At this point, if you want to teach your students about planar affine geometry, this is a good place to introduce them to the axioms. (We didn't do this with the high schoolers, mostly because we didn't have enough time; we have done it with college students in a geometry class.) The axioms for planar affine geometry can be found in a number of places, for example, [4]. We have chosen the following axioms.

1. Axiom 1: There are at least three non-collinear points. (Translation: There are at least three cards that do not form a set. This axiom guarantees that there are enough points and lines to make the geometry interesting.)
2. Axiom 2: Given any two distinct points P and Q, there is exactly one line passing through them. (Translation: Two cards determine a unique set. This is what we have called the Fundamental Theorem of SET®.)
3. Axiom 3: Given any line ℓ and a point P not on ℓ, there is exactly one line through P parallel to ℓ. (Translation: Given a set and a card not in the set, there's a unique parallel set.)

What does parallel mean in the context of SET®? You can ask students to look at parallel sets in their configuration containing cards from the entire deck and to characterize when sets are parallel. One example is shown in Figure 5.

Students should be able to recognize (perhaps with nudging) that if one set has its symbols all the same for a specific attribute, then the parallel set will also have all symbols the same for that attribute, though it may be different expression of that attribute. For example, in Figure 5, all the symbols in the top set are solid, and all symbols in the bottom set are striped. When multiple attributes are different, then the cards in the two sets can be lined up so that

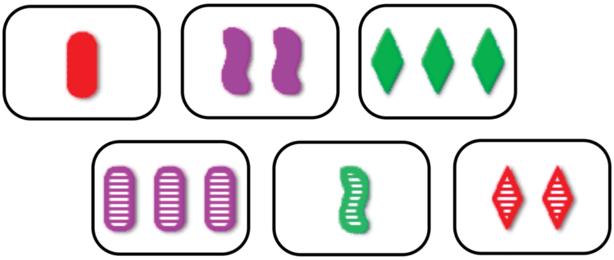


Figure 5. Two parallel sets.

the symbols cycle through the expressions for each attribute in the same way, moving from left to right. Notice that in Figure 5, the cards in the two sets are lined up so the shapes are in the same order. The symbols' numbers and colors now cycle in the same way moving left to right: one, two and three and red, purple, and green. When two sets are not parallel, you may have one attribute cycling the same way, but another attribute will not, so no matter how you line the cards up, you cannot get the attributes to cycle the same way. (One consequence: any two sets that differ only in a given attribute will always be parallel.)

Now, students can choose a set and another card from the deck, and find the set parallel to the first set containing the card. For example, take either of the two sets in Figure 5, and consider the card with two empty red squiggles. The next card in the set parallel to the given sets must have empty symbols, since that attribute must all be the same. Now following the cycling of the other attributes, you can see that the next card in the parallel set must be the card with three empty purple diamonds. You can now complete the set with the card that has one empty green oval. You can ask students to explain why this parallel set is unique. Furthermore, they can line up their two sets and make another magic square with those two sets as the first two lines in an affine plane of order 3, helping them to see the uniqueness of the magic square as well.

Students can now extend the geometry to three dimensions by building a cube. An example of cards showing the three-dimensional affine geometry is shown in Figure 6.

Start with a plane of cards and then choose another card. Find the horizontal line through the card parallel to the horizontal lines in the plane, then find the vertical line through the card parallel to the vertical lines in the plane,

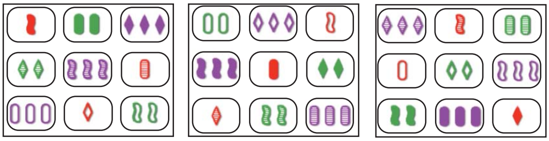


Figure 6. The three-dimensional geometry $AG(3, 3)$.

then finish the plane as was done in Figure 4. Now, you can fill in the third plane by completing the sets in each position. For example, the two cards in the upper left of the first two planes are one solid red squiggle and two empty green ovals. Thus, the card in the upper left of the third plane will be the card that completes that set, three striped purple diamonds.

Notice that in this configuration, any set is either all in the same subplane or has one card in each subplane; furthermore; it occupies positions in each of the subplanes that correspond to a set in $AG(2, 3)$ (or occupies the same position in each subplane). Thus, just as we saw that three non-collinear cards (cards that don't make a set) determine a magic square, four non-coplanar cards (cards that cannot be put in the same plane) will determine a magic cube.

Now, your students (and you!) are ready to lay the entire deck out in a four-dimensional configuration. Start with a magic cube. Choose any card that remains in the deck. You can complete a subplane containing that card in the same way you did before by building sets parallel to corresponding lines in the subplanes above, and put it below the left-most subplane; then fill in the third subplane that completes the three-dimensional geometry as before. You can make the remaining subplanes in the same way: the one in the middle of the configuration will have the cards that complete sets with the upper right subplane and the lower left subplane. An example is shown in Figure 7. Notice that the three-dimensional cube from Figure 5 is the hyperplane at the top of

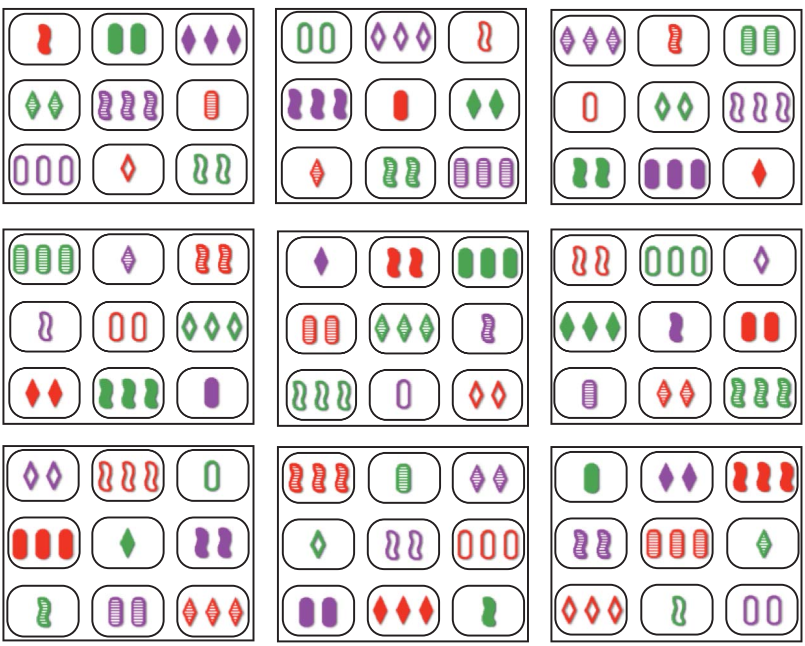


Figure 7. The full deck of cards organized as the four-dimensional geometry $AG(4, 3)$.

the geometry. Notice also that: two cards determine a set, three non-collinear cards determine a magic square, four non-coplanar cards determine a magic cube, and finally, five non-co-cubic cards determine the entire geometry! Any set will now either be in one subplane or in three subplanes in the same position as points in line in $AG(4, 3)$.

Students really enjoy making this configuration. One college student made one on the floor in her dorm room, and then wouldn't let her roommate walk near it for several days! And though we only asked the high schoolers to make the three-by-three configurations, a few of them had the idea to keep going and ended up laying out the entire deck on their own. It was incredibly rewarding to watch them create the magic hyper-cube all by themselves.

4. LINEAR ALGEBRA

We can now introduce numerical labels, which will put us in the realm of linear algebra. Viewing the cards this way will allow us to learn even more about the game, which is a wonderful way of reinforcing how useful linear algebra is. Assign the numbers 0, 1, and 2 to correspond to each possibility of each attribute. It doesn't matter how you do this, but once you've chosen your numbering, you need to keep it fixed. Table 1 is one example of such a numbering correspondence.

Then, choose a set, and ask the students to find the coordinates for the cards in the set. One example, given the numbering scheme above, is shown in Figure 8.

Table 1. The numbering correspondence

	Attribute		
	0	1	2
Number	3	1	2
Color	green	red	purple
Shading	empty	striped	solid
Shape	diamond	oval	squiggle

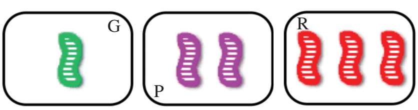


Figure 8. The coordinates for these cards are (1,0,1,2), (2,2,1,2), and (0,1,1,2) (color figure available online).

A set is three cards whose coordinatewise sum is (0,0,0,0) modulo 3. ($0+0+0 \equiv 1+1+1 \equiv 2+2+2 \equiv 1+2+3 \equiv 0$ mod 3, so requiring an attribute to be all the same or all different guarantees that the mod 3 sum for that coordinate will be zero.) This fact by itself is enough to guarantee that the deck of SET® cards is indeed $AG(4, 3)$. You can ask students to see if they can use vectors to describe when two sets are parellel. In fact, two sets are parallel if you can find a vector that you can add to each card in one set to give the other set. This nicely dovetails with the vector formulation of lines in $\mathbb{R}^3$.

We can now use affine transformations to understand the geometry better. An *affine transformation* is a linear transformation followed by a translation. The linear transformation needs to be full rank, so that the collection of cards is preserved, and the translation is needed because linear transformations always fix the $\vec{0}$ vector. The $\vec{0}$ vector is just whichever card has coordinates (0,0,0,0), which isn't particularly special to us. This corresponds to the fact that in affine geometry, there is no distinguished point (or origin). We can write the affine transformation as $\vec{x} \mapsto \mathbf{A}\vec{x} + \vec{b}$ for some 4×4 matrix $\mathbf{A}$ of rank 4 and some 4-vector $\vec{b}$.

Consider the two planes shown in Figure 9 containing the same cards, so they don't really give us any new information. Ask the students to find an affine transformation that takes the first plane to the second.

There are many solutions; one possible example is:

$$\vec{y} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

It is nice to let students work this out, since they will see how much freedom they have. Another good exercise would be to have students make a plane, and choose a 4×4 matrix $\mathbf{B}$ of full rank and a vector $\vec{c}$. They can then apply the affine transformation $\vec{x} \mapsto \mathbf{B}\vec{x} + \vec{c}$ for each card and see which cards they get.

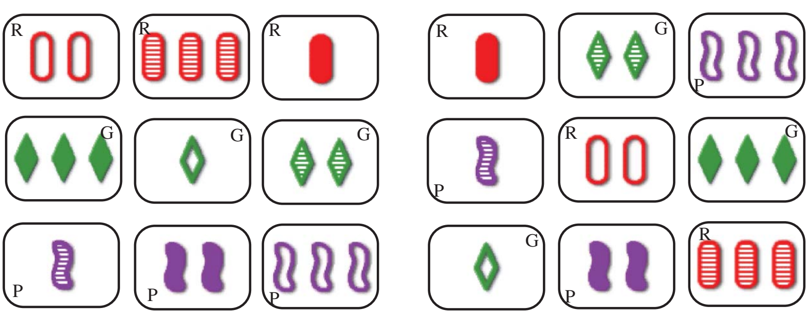


Figure 9. Two planes containing the same cards (color figure available online).

Indeed, they will have made a new plane. This shows that full rank (invertible) affine transformations preserve lines (sets) and planes; by extension, they also preserve hyperplanes. The applet Swingset, developed by Coleman, Hartshorn, Long and Mills and available at www.mathcs.moravian.edu/~swingset/ [1] provides a nice way to visualize these affine transformations.

At this point, you could explore the full group of symmetries of the geometry *AG*(4, 3), for students who know some group theory. The full symmetry group of *AG*(*n*, 3) is the affine group $Aff(n, 3) = GL(n, 3) \times \mathbb{Z}_3^n$ (so named because it's the symmetry group of the affine space). Wikipedia provides good information on this group [5].

5. THE END OF THE GAME

Here is an amazing SET® fact: If you remove one of the cards at the beginning of the game without looking at it and put it aside face down, you can *determine the missing card from the cards left on the table at the end*. (When you introduce this, it will look like a magic trick, so by all means bask in that glory for as long as possible before explaining why it works.) We call this End Game SET®, and a description of the game can be found on the SET® website [3], (under Teacher's Corner, Browse by game, Set, End Game SET).

Here's how to play: Put one card aside at the beginning of the game, face down. At the end of the game when there are no more sets left on the board, mentally remove single-attribute "sets" by considering only one attribute at a time. (When you begin this, you can let the students physically rearrange the cards to help them see the single-attribute "sets," but doing it mentally looks more impressive.) Suppose the eight cards in Figure 10 remain at the end of the game (with one card hidden). Let's determine the missing card.

In Figure 11, at the top, after removing the two indicated "sets," the two leftover cards are both striped, so the last card must be striped. Likewise, using the next two configurations in Figure 11, we can determine that the missing shape is diamonds, the color must be purple, and the number must be two. So the final card must be two striped purple diamonds. (Actually, you can group the "sets" any way you want and end up with the same result; we've just shown one example of how you might do this.)

Now comes the impressive part. Take another look at the cards in Figure 10. Can you see any cards that would make a set with two striped purple diamonds? Yes, you can! The final solution can be seen in Figure 12 with our missing card put back in.

The winner of this part of the game is the person who can figure out what the hidden card is and *without turning it over* call "Set!", pull out two of the cards, turn over the hidden card, and display the set formed. You will not always be able to make a set with the last card and the final layout, but you can always determine what that last card is. If there is no set, the winner is the first to

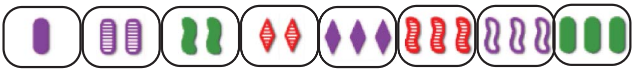


Figure 10. The final cards with no sets left.

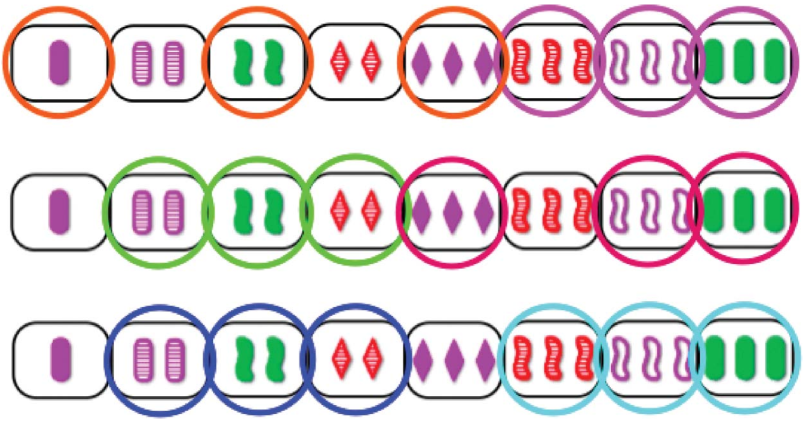


Figure 11. Top: Removing by shading. Middle: Removing by shape. Bottom: Removing by color and number; in this case you can consider two attributes at once.

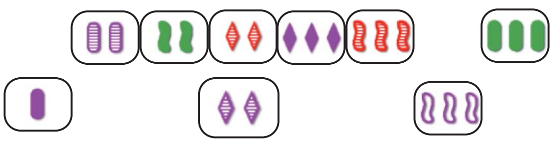


Figure 12. SET!!

determine the card and decide that there is no set. Either way, you can appear to be magical when you show this trick off for the first time.

There is a lovely explanation of why the End Game works that uses modular arithmetic. As we saw in Section 4, a set is three cards whose coordinatewise sum is (0,0,0,0) modulo 3. We can easily see that the 81 cards can be partitioned any number of ways into 27 sets; one example was shown in Figure 7 in Section 3. Thus, the coordinatewise sum of the entire deck should also be (0,0,0,0) modulo 3. (Alternatively, there are 27 cards with each particular form of an attribute. Thus, for that coordinate, those cards will sum to zero.) Now, when we reach the end of the game, we have a stack of sets whose coordinatewise sum is (0,0,0,0) mod 3, and some number of cards left over that is a multiple of three: those leftover cards better sum to (0,0,0,0) mod 3 as well. (In fact, this is a great parity check: if the End Game doesn't work, then someone made a mistake when they took a set!) This is why we can isolate each individual attribute and make single-attribute "sets": we are actually doing modular arithmetic in disguise on each coordinate. Furthermore, this proves that we cannot have three cards left over at the end: if we did, they would be a set. Thus, if you ever get down to the last six cards and find a set, go ahead and call "Set!" twice, and take all six cards. In our years of playing SET®, this has happened maybe a dozen times.

6. CONCLUSIONS

Students (and faculty!) really enjoy the game of SET®, and there is a mountain of mathematics to be found within it. In fact, this article only gives a brief glimpse of what you can find. For additional reading, there's an excellent article from the *Mathematical Intelligencer* by Davis and Maclagan [2]. There is also quite a bit of information about how the game was invented (by Marsha Jean Falco, a population geneticist) as well as other information on the SET® website, [3], where you can see the early versions of the game. We have found that our students get quite involved in the game, and their enthusiasm often extends past the time the game is discussed. We encourage you to let your students explore even more. There's a great deal to be found.

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BIOGRAPHICAL SKETCHES

Hannah Gordon received her B.A. and M.A.T. from the University of Chicago in mathematics, music, and mathematics education, respectively. She is currently a math teacher with the Chicago Public Schools. She also enjoys singing, cooking, running, and writing brief autobiographical sketches. She hopes one day to beat aliens in a game of SET® and thus earn the title SET® Grand Master of the Universe.

Rebecca Gordon received her education at Oberlin College and Victoria University of Wellington in English Literature before deciding math was more fun than analytical literature. She taught high school math for a year at a girl's school in New Zealand and is continuing her teaching career at Newark Academy in Livingston, NJ. She enjoys reading, rock climbing, and baking, and she plays the banjo.

Elizabeth McMahon earned degrees from Mount Holyoke, the University of Michigan and UNC. Her mathematical interests are in combinatorics, algebra and SET®. She enjoys biking to work when weather permits, hiking, travel, and being crushed by her daughters in SET® (she's pretty good, though). She loves collaborating with them, and with their father, Gary Gordon.