ELSEVIER

Discrete Mathematics 232 (2001) 19–33

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

A characteristic polynomial for rooted graphs and rooted digraphs

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Received 4 February 1999; revised 27 March 2000; accepted 10 April 2000

Abstract

We consider the one-variable characteristic polynomial $p(G;\lambda)$ in two settings. When G is a rooted digraph, we show that this polynomial essentially counts the number of sinks in G. When G is a rooted graph, we give combinatorial interpretations of several coefficients and the degree of $p(G;\lambda)$. In particular, $|p(G;0)|$ is the number of acyclic orientations of G, while the degree of $p(G;\lambda)$ gives the size of the minimum tree cover (every edge of G is adjacent to some edge of T), and the leading coefficient gives the number of such covers. Finally, we consider the class of rooted fans in detail; here $p(G;\lambda)$ shows cyclotomic behavior.

Keywords: Characteristic polynomial; Rooted graph; Rooted digraph; Branching greedoid

1. Introduction

Rooted graphs and digraphs are important combinatorial structures that have wide application, but they have received relatively little attention from the viewpoint of graphic invariants. A fundamental reason for this oversight is that although the Tutte polynomial, characteristic polynomial, β-invariant, and other invariants have been well-studied for ordinary graphs and matroids, rooted graphs and digraphs do not have a matroidal rank function.

In spite of this deficiency, rooted graphs and digraphs do have 'natural' rank functions which impart a greedoid structure to the edge set. The resulting objects are called *branching greedoids* (in the case of rooted graphs) and *directed branching greedoids* (in the case of rooted digraphs). Applying the tools developed in [6–8] for greedoid invariants allows a meaningful application to rooted graphs and digraphs.

We note that one and two variable polynomials for digraphs have been considered in other contexts. Chung and Graham [5] develop a Tutte-like polynomial for non-rooted

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0012-365X/01/$ - see front matter
PII: S0012-365X(00)00186-2

digraphs which has several interesting invariants among its evaluations. Several different polynomials for rooted digraphs are described in [9]; the various polynomials are related, but they highlight differing aspects of the rooted digraph.

Our goal in this paper is to continue the investigation of the characteristic polynomial begun in [7], concentrating exclusively on rooted graphs and digraphs. Like many important areas of combinatorics, the development of the characteristic polynomial traces its origin to attempts to solve the 4-color problem. Chromatic polynomials for graphs, introduced in such attempts, were subsequently generalized to matroid characteristic polynomials [17]. These polynomials share many of the attractive properties that chromatic polynomials have and count several interesting invariants, especially when the matroid is represented over a field.

There are several ways to judge the effectiveness of the generalization of an invariant in combinatorics:

- Do standard results remain true in the new setting?
- Are there reasonable combinatorial interpretations for the invariant?
- Does the invariant exhibit interesting behavior in the new setting?
- Do the techniques generate new combinatorial results which might be difficult to prove (or even discover) otherwise?

We will see that $p(G)$ will satisfy all of these criteria at some level. We believe the results given here motivate continued study of the characteristic polynomial for these and other greedoids.

The paper is organized as follows: Section 2 summarizes the basic results about $p(G)$ which we will require. Section 3 considers the characteristic polynomial $p(D;\lambda)$ when D is a rooted digraph. In this case, the polynomial is especially simple, dependent only on the number of sinks in the digraph D (Theorem 3.4).

In Section 4, we prove several general results for $p(G;\lambda)$ when G is a rooted graph. The main results of this section are the combinatorial interpretations of the degree of $p(G;\lambda)$ (Theorem 4.6), the leading coefficient (Corollary 4.7), and $p(G;0)$ (Theorem 4.8), the last of which is essentially equivalent to a theorem of Greene and Zaslavsky [10] on acyclic orientations. (Although we use the result of Greene and Zaslavsky to prove Theorem 4.8, it is easy to construct an independent proof.)

Finally, in Section 5 we examine one class of rooted graphs in detail. We concentrate on rooted fans and investigate the factoring properties of $p(F_n;\lambda)$. Fans are an important class of graphs which have been studied in reference to minimally 3-connected graphs [14,16]. They also occur naturally as minors of wheels, another important class of graphs. When G is a rooted fan, $p(F_n;\lambda)$ factors (over the rationals) in a manner essentially equivalent to that of x^n-1 into 'cyclotomic pieces'. As an application, we compute the number of *minimum tree covers* (subtrees T of the fan F_n in which every edge of F_n is adjacent to some edge in T) via the polynomial. The proof uses elementary properties of the polynomial and a recursion.

2. Definitions and fundamental properties

We begin with some basic definitions. For more information on greedoids, see [1] or [12].

Definition. A *greedoid* G on the ground set E is a pair $(E,\mathscr{F})$ where E is a finite set and $\mathscr{F}$ is a family of subsets of E (called the *feasible sets*) satisfying

1. For every non-empty $X \in \mathscr{F}$ there is an element $x \in X$ such that $X - \{x\} \in \mathscr{F}$;
2. For $X, Y \in \mathscr{F}$ with $|X| < |Y|$, there is an element $y \in Y - X$ such that $X \cup \{y\} \in \mathscr{F}$.

The *rank* of a subset A of E, denoted $r(A)$, is defined to be the size of the largest feasible subset of A, i.e.

$$r(A) = \max_{S \in \mathscr{F}} \{|S|\colon S \subseteq A\}.$$

An element e of the ground set of G is a *greedoid loop* if e is in no feasible set.

A rooted graph G with distinguished vertex $*$ satisfies this definition if the ground set E is the edge set of G and if the feasible sets $\mathscr{F}$ are the rooted subtrees F of G. The greedoid associated with G is called the *branching greedoid*. The greedoid loops of G are edges which join a vertex to itself (ordinary loops), and any edge that is not in the same component as the root.

A rooted digraph D with distinguished vertex $*$ also satisfies this definition if the ground set E is the edge set of D and if the feasible sets $\mathscr{F}$ are the rooted arborescences F of D (i.e., F contains the root $*$ and, if v is a vertex in F, there is a unique directed path in F from $*$ to v). This is the *directed branching greedoid* associated with D. The greedoid loops are precisely the edges $e = vw$ (having initial vertex v and terminal vertex w) that are in no rooted arborescences, i.e., edges where $v = w$ (ordinary loops, as before), edges with $w = *$, edges with the property that every directed path from $*$ to v passes through w, and edges that are inaccessible from $*$.

We will use an evaluation of the 2-variable Tutte polynomial of a greedoid to define the characteristic polynomial of a greedoid.

Definition. Let G be a greedoid on the ground set E. The *Tutte polynomial* of G is defined by

$$f(G;t,z) = \sum_{S \subseteq E} t^{r(G)-r(S)} z^{|S|-r(S)}.$$

This polynomial was introduced in [6], and has been studied for various greedoid classes. A deletion-contraction recursion (Theorem 3.2 of [6]) holds for this Tutte polynomial, as well as an activities expansion (Theorem 3.1 of [8]).

Proposition 2.1 (Gordon and McMahon [8, Theorem 4.2]). *Let D be a rooted digraph with no greedoid loops. If $f(D;t,z) = (z+1)^k f_1(t,z)$, where $z+1$ does not divide*

$f_1(t,z)$, then k is the minimum number of edges that need to be removed from D to leave a spanning, acyclic rooted digraph.

Proposition 2.2 (McMahon [13, Theorem 2]). *Let G be a rooted graph with $f(G;t,z)=(z+1)^a f_1(t,z)$, where $z+1$ does not divide $f_1(t,z)$. Then a is the number of greedoid loops in G.*

The characteristic polynomial for greedoids was defined in [7].

Definition. Let G be a greedoid on the ground set E. The *characteristic polynomial* $p(G;\lambda)$ is defined by

$$p(G;\lambda)=(-1)^{r(G)}f(G;-\lambda,-1).$$

Here are some of the results we will need for the characteristic polynomial:

Proposition 2.3 (Boolean expansion; Gordon and McMahon [7, Proposition 1]).

$$p(G;\lambda)=\sum_{S\subseteq E}(-1)^{|S|}\lambda^{r(G)-r(S)}.$$

Proposition 2.4 (Deletion-contraction; Gordon and McMahon [7, Proposition 3]). *Let $\{e\}$ be a feasible set in G. Then*

$$p(G;\lambda)=\lambda^{r(G)-r(G-e)}p(G-e;\lambda)-p(G/e;\lambda).$$

Proposition 2.5 (Direct sum property; Gordon and McMahon [7, Proposition 4]).

$$p(G_1\oplus G_2)=p(G_1)p(G_2).$$

Proposition 2.6 (Gordon and McMahon [7, Proposition 5]). $(\lambda-1)|p(G)$.

Note that we could equally well take the definition of $p(G;\lambda)$ from either of Propositions 2.3 or 2.4.

We will need one more expansion for $p(G;\lambda)$, an expansion in terms of feasible sets. In [8], we develop a notion of external activity for feasible sets. See the discussion preceeding Proposition 2 in [7] for more details. Briefly, a *computation tree T_G* for a greedoid G is a recursively defined, rooted, binary tree in which each node of T_G is labeled by a minor of G. At each stage we label the two children of a node corresponding to a minor H by $H-e$ and H/e, where $\{e\}$ is a feasible set in H. Then there is a bijection between the feasible sets of G and the terminal vertices of the computation tree; the feasible set is simply the set of elements of G which are contracted in arriving at the specified terminal node. The *external activity of a feasible set F with respect to the tree T_G* is the collection of elements of G which were neither deleted nor contracted, that is, the greedoid loops which remain at that leaf of the computation tree.

Proposition 2.7 (Feasible set expansion; Gordon and McMahon [7, Proposition 2]). *Let T_G be a computation tree for G and let $\mathscr{F}_{\mathscr{T}}$ denote the set of all feasible sets of G having no external activity. Then*

$$p(G;\lambda) = \sum_{F\in\mathscr{F}_{\mathscr{T}}} (-1)^{|F|}\lambda^{r(G)-|F|}.$$

3. Rooted digraphs

In this section, we completely determine the characteristic polynomial for rooted digraphs: we show that $p(D;\lambda) = (-1)^{r(D)}(1-\lambda)^s$, where s is the number of sinks in an acyclic disgraph D.

Lemma 3.1. *Suppose D is a rooted digraph with a directed cycle. Then* $p(D;\lambda) = 0$.

Proof. This result follows from Proposition 2.1 and the definition of the characteristic polynomial in Section 2. □

Thus, we may assume D is acyclic. Recall that if e is an edge, then the set $\{e\}$ is *feasible* if e is adjacent to the root and is directed away from the root. The next result is immediate.

Lemma 3.2. *Let D be a rooted digraph consisting of a single feasible edge. Then* $p(D) = \lambda - 1$.

The deletion/contraction algorithm for rooted digraphs can be performed in a more efficient way than the general recursion (Proposition 2.4) can.

Proposition 3.3. *Suppose e is a feasible edge of an acyclic rooted digraph D, where e is not a leaf.*

1. *If e is in every basis, then* $p(D;\lambda) = -p(D/e;\lambda)$.
2. *If e is not in every basis, then* $p(D;\lambda) = p(D-e;\lambda)$.

Proof. Suppose D is a rooted digraph with no directed cycles and e is a feasible edge of D which is not a leaf.

1. Suppose e is in every basis. Since e is not a leaf, there must be at least one edge of D that is only in feasible sets that contain e. (Suppose there is no such edge. Then every feasible pair of edges $\{e,f\}$ must have $\{f\}$ feasible as well. In this case, e is a leaf.) Thus, $D-e$ has a loop, so $p(D-e;\lambda) = 0$. Hence, from Proposition 2.4, $p(D;\lambda) = -p(D/e;\lambda)$.

2. Suppose there is a basis B that does not contain e, and let v be the terminal vertex of e. There must be another edge f that also has v as its terminal vertex since B is a

basis, and every basis must reach v. In this case, however, f is a loop in D/e, since the terminal vertex of f will be $*$ in D/e. Thus, $p(D/e;\lambda)=0$. Because f has the same terminal vertex as e, and there must be a path from $*$ ending in f, the terminal vertex of e is reachable in $D-e$, so $r(D-e)=r(D)$. Thus, $p(D;\lambda)=\lambda^{r(D)-r(D-e)}p(D-e;\lambda)-p(D/e;\lambda)=p(D-e;\lambda)$. □

Remark. This proposition tells us that we can compute the characteristic polynomial for D in a particularly simple way. We may begin with D and choose any feasible edge e which is not a leaf. We then either delete or contract e, as indicated by the proposition. Eventually, we arrive at a greedoid minor consisting of leaves only. The following theorem completes the picture.

Theorem 3.4. *Let D be a rooted digraph. If D contains no greedoid loops and no directed cycles, then $p(D;\lambda)=(-1)^{r(D)}(1-\lambda)^s$, where s is the number of sinks in D.*

Proof. First, note that if a digraph D has no directed cycles, then it must have at least one sink. Assume that D is a rooted digraph with no greedoid loops and no directed cycles. We compute $p(D)$ via Proposition 3.3, by choosing feasible edges which are not leaves and either deleting or contracting. Let D' be a digraph obtained from D by repeated application of Proposition 3.3. Then a feasible edge e of D' is deleted if it is not in every basis of D', which occurs exactly when there is another path from $*$ to the terminal vertex of e. In this case, there is no factor of -1 introduced, i.e., $p(D')=p(D'-e)$. On the other hand, a feasible edge e is contracted if that edge is in every basis in D', which occurs exactly when there is no other path to the terminal vertex of e. In this case, contracting e will have the effect of reducing the rank of D by 1 and introducing a factor of -1, i.e., $p(D')=-p(D'/e)$.

The process of deleting and contracting edges terminates when only leaves remain. The terminal vertex of a leaf at this point corresponds precisely to a sink in the original digraph D. By the direct sum property and Lemma 3.2, each leaf will contribute a factor of $(-1)(1-\lambda)$. Finally, a factor of -1 is introduced for each single rank drop. Putting these pieces together gives the formula. □

Although an inductive proof of this result follows from Proposition 3.3, we prefer the proof given above, which highlights the connection between the recursive procedure of the Proposition and sinks in D.

4. Rooted graphs

In this section, we concentrate on the characteristic polynomial for rooted graphs. In general, there is no analog to Theorem 3.4 for rooted graphs in the sense that there is no known simple formula that describes the graph theoretic information encoded in $p(G;\lambda)$. Our main results here (Theorems 4.6 and 4.8) show how to interpret the degree of the polynomial and the evaluation $p(G;0)$ combinatorially.

It is easy to determine $p(G;\lambda)$ for some special cases. We omit the straightforward proofs of the next proposition.

Proposition 4.1. (1) *Let T be a rooted tree with n edges and l leaves. Then* $p(T;\lambda)=(-1)^{(n-l)}(\lambda-1)^l$.

(2) *Let C be a rooted cycle with n edges. Then* $p(C;\lambda)=(-1)^n(n-1)(\lambda-1)$.

(3) *Let K_n be the rooted complete graph on n vertices* (*including the root*). *Then* $p(K_n;\lambda)=(-1)^n(n-1)!(\lambda-1)$.

We will need a few results which simplify the calculation of $p(G;\lambda)$. Propositions 4.2, 4.4, and 4.5 are consequences of the definition of the characteristic polynomial and Proposition 2.2. These results will allow us to restrict our attention to rooted graphs with no greedoid loops and no multiple edges; further, because of the direct sum property, we need only delete and contract edges which are not leaves.

Feasible edges which are leaves are greedoid *isthmuses*, that is, edges that can be added to or deleted from any feasible set without affecting feasibility. If e is an isthmus in G, then G is the direct sum (as a greedoid) of G/e with the one-element greedoid on $\{e\}$. Every isthmus is also a (greedoid) *coloop*, i.e., an edge which is in every basis. For matroids, these two notions coincide, although clearly they are different in greedoids.

Proposition 4.2. *Suppose G is a rooted graph with no greedoid loops. Let $\bar{G}$ be the corresponding graph with all multiple edges replaced by single edges. Then* $p(G)=p(\bar{G})$.

Proof. In deleting and contracting edges, any multiple edges will simply be carried along until they are adjacent to $*$. Suppose $\{e_1,e_2,\ldots,e_n\}$ is a set of edges in G, each of which joins $*$ and another vertex v. By Proposition 4.4, $p(G;\lambda)=p(G-e_1;\lambda)-p(G/e_1;\lambda)$. Now e_i (for all $i\geqslant 2$) is a greedoid loop in G/e_1, so $p(G/e_1)=0$. Hence, $p(G;\lambda)=p(G-e_1;\lambda)$, and we can continue to delete these multiple edges in turn until only one remains. In other words, those edges could simply have been deleted at the beginning and the polynomial would be the same. □

The next proof is a straightforward calculation.

Lemma 4.3. *Suppose G is a rooted graph consisting of a single leaf. Then* $p(G;\lambda)=\lambda-1$.

Thus, by the direct sum property (Proposition 2.5), if G has an isthmus e, then $p(G)=(\lambda-1)p(G/e)$.

We now consider what happens if a feasible edge which is not a leaf is deleted or contracted.

Proposition 4.4. *Let G be a rooted graph with no greedoid loops. If* $\{e\}$ *is feasible, but e is not a leaf, then*

$$p(G;\lambda) = p(G-e;\lambda) - p(G/e;\lambda).$$

Proof. Suppose e is a feasible edge which is not a leaf. From Proposition 2.4, $p(G;\lambda)=\lambda^{r(G)-r(G-e)}p(G-e;\lambda)-p(G/e;\lambda)$. If $r(G)=r(G-e)$, then we are done. On the other hand, if $r(G)\neq r(G-e)$, then there is no path from $*$ to the terminal vertex of e in $G-e$. Since e is not a leaf of G, this means that there must be a greedoid loop in $G-e$, so $p(G-e)=0$. Hence, $p(G;\lambda)=p(G-e;\lambda)-p(G/e;\lambda)$. □

Proposition 4.5. *Suppose G is a rooted graph with no greedoid loops, and e is a coloop but not an isthmus. Then* $p(G;\lambda)=-p(G/e;\lambda)$.

Proof. This proof is the same as the case for rooted digraphs, in Proposition 3.3. □

We are now ready for the main results of this section. Define a *minimum tree cover* for G to be a minimum size feasible set T with the property that every edge of G is adjacent to some edge of T. A minimum tree cover T has the property that T is a minimum size subtree such that every edge of the minor G/T is adjacent to $*$.

Theorem 4.6. *If G is a rooted graph, let* $a(G)$ *be the size of a minimum tree cover for G. Then the degree of the characteristic polynomial* $p(G;\lambda)$ *is equal to* $r(G)-a(G)$.

Proof. Let $\mathscr{M}$ be the family of all minimum tree covers for G. Now let T_G be a computation tree for G and let F be a minimum size feasible set with no external activity, so that the contribution of F to the feasible set expansion (Proposition 2.7) is $\pm\lambda^k$ and k is maximum.

We wish to show that $F\in\mathscr{M}$. Because F has no external activity with respect to the computation tree T_G, every edge e of G which is not in F was deleted; hence e was adjacent to the root $*$ at the time it was deleted. Now if e is adjacent to $*$ in G, e must be adjacent to some edge of F, since F is feasible and cannot be empty. If e is not adjacent to $*$ in G, then e became adjacent to $*$ when some edge f of F was contracted, so e is adjacent to f in G. In either case, every edge e is adjacent to some edge in F.

Now suppose that there is a feasible set F' with $|F'|<|F|$ such that every edge of G is adjacent to some edge of F'. Since F was a minimal size feasible set with no external activity in T_G, then F' must have had external activity. We will create a new computation tree T'_G such that the edges of F' are contracted so that no greedoid loops are created. This is always possible because if the contraction of an edge $e\in F'$ would create a loop e', then e' must have been adjacent to $*$ when e was contracted (because

F' is adjacent to every edge of G). Thus, e' can be deleted *before* e is contracted in the computation tree. Thus, the terminal vertex in T'_G which corresponds to F' has no greedoid loops, so F' has no external activity. Thus, F' will contribute $\pm\lambda^s$ to $p(G)$ where $s > k$. However, the polynomial is independent of the computation tree, so F could not have been a minimal size feasible set with no external activity in the original computation tree. This is a contradiction, hence $F \in \mathscr{M}$. Hence, the degree of the characteristic polynomial $p(G;\lambda)$ is equal to $r(G) - a(G)$, as desired. □

Corollary 4.7. *Suppose* $\deg(p(G;\lambda)) = k$ *and let* a_k *be the coefficient of* λ^k. *Then* $(-1)^{r(G)-k}a_k$ *is the number of minimum tree covers of* G.

The next result is closely related to the remarkable fact (discovered by Greene and Zaslavsky [10]) that the number of acyclic orientations of a graph with a unique (specified) source is independent of that source. Our proof uses their result.

Theorem 4.8. *Let* G *be a rooted graph with root* $*$. *Let* $\mathscr{O}(G)$ *be the collection of all acyclic orientations of* G *with a unique source* $*$. *Then* $|\mathscr{O}| = (-1)^{r(G)}p(G;0)$.

Proof. If G is a rooted graph, let $\bar{G}$ be the unrooted graph that is obtained when the root of G is treated as any other vertex. Let $h(\bar{G};\lambda)$ be the matroid characteristic polynomial for graphs (see [17]). (The definition can be taken to be precisely the same as that of Proposition 2.3, where the rank function $r(A)$ is the size of the largest acyclic subset of A.)

Greene and Zaslavsky [10] proved that $(-1)^{r(G)}h(\bar{G};0)$ equals the number of acyclic orientations of G with unique source $*$. (Since the calculation of $h(\bar{G};\lambda)$ does not depend on $*$, this shows that the number of acyclic orientations of a graph with a unique (specified) source is independent of that source.) Thus, we can finish our proof by showing $p(G;0) = h(\bar{G};0)$.

Note first that if e is a greedoid loop in G, then either (1) e is an ordinary loop (which is a cycle) or (2) e is disconnected from the root $*$ (so G has more than one component). In either case, there are no acyclic orientations of G in which $*$ is the unique source. Since $p(G;\lambda) = 0$ in this case, the result holds.

Thus, we may assume G has no greedoid loops. If every edge of G is a feasible leaf (a greedoid isthmus), then $p(G;\lambda) = (\lambda - 1)^k$ for some k. There is only one legal acyclic orientation in this case, so the result follows.

Finally assume that e is feasible, not a leaf, and G has no greedoid loops. By Proposition 4.4, we have $p(G;\lambda) = p(G-e;\lambda) - p(G/e;\lambda)$. We now complete the proof by induction on the number of edges of G. The base cases are handled above. Now induction yields $p(G-e;0) = h(\bar{G}-e;0)$ and $p(G/e;0) = h(\bar{G}/e;0)$. But $h(\bar{G};\lambda)$ satisfies the same recursion when e is not an isthmus (Theorem 4.2 of [3]): $h(\bar{G};\lambda) = h(\bar{G}-e;\lambda) - h(\bar{G}/e;\lambda)$. Hence

$$h(\bar{G};0) = h(\bar{G}-e;0) - h(\bar{G}/e;0) = p(G-e;0) - p(G/e;0) = p(G;0). \qquad \square$$

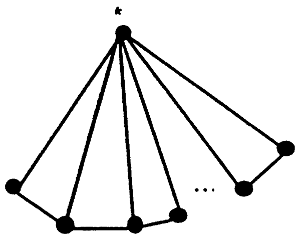

Fig. 1. The fan F_n.

It is interesting to note that since the number of acyclic orientations of G with unique source $*$ is independent of the choice of $*$, so is the calculation $p(G;0)$, i.e., $p(G;0)$ does not depend on the choice of $*$. It would be interesting to compare the full polynomials $h(\bar{G};\lambda)$ and $p(G;\lambda)$ in more detail.

We also remark that an inductive proof of Theorem 4.8 which does not explicitly refer to Greene and Zaslavsky's result is not difficult. This is essentially the approach in Theorem 6.3.18 of [4].

It is also worth noting that if $\mathcal{O}$ is an acyclic orientation of a rooted graph G, then $*$ is the unique source in $\mathcal{O}$ if and only if there are no greedoid loops in the directed branching greedoid associated with $\mathcal{O}$. Thus, we can restate Theorem 4.8 as follows:

Corollary 4.9. *Let G be a rooted graph with root $*$ and let $\mathcal{O}(G)$ be the collection of all acyclic orientations of G which create no greedoid loops in the directed branching greedoid associated with $\mathcal{O}$. Then $|\mathcal{O}| = (-1)^{r(G)}p(G;0)$.*

5. Rooted fans

We conclude with a careful treatment of one important class of graphs, rooted fans F_n (see Fig. 1). These graphs arise in the study of non-essential edges in 3-connected graphs [14,16], as well as in other areas. In particular, they model distribution systems in which there is a central node (the root) that is adjacent to all of the remaining nodes, which, in turn, are joined by a simple path. For example, this could be the arrangement of a satellite broadcasting system, where the satellite can communicate directly with a linear arrangement of ground stations.

Understanding the characteristic polynomial for this class is also important. Applying Theorems 4.6–4.8 to rooted fans gives new combinatorial information about this class (see the remark following the proof of Proposition 5.3). Our main result (Theorem 5.7) gives a complete factorization of $p(F_n)$ over the rationals which is closely connected to the factorization of $x^n - 1$.

Generally, factorization questions involving the chromatic and characteristic polynomials are of wide interest. Stanley's modular factorization theorem (Theorem 2 of [15]) shows why factoring the characteristic polynomial of a combinatorial geometry (simple matroid) gives information about the structure of the geometry, and Brylawski's theorem on parallel connections of matroids (Theorem 6.16(v) of [2]) shows that the characteristic polynomial is essentially multiplicative on parallel connections. The results we develop here fit into this context.

We will need several preliminary results to derive the formulas we will use. We use the convention that F_n has $n+1$ vertices (and $2n-1$ edges).

Our first result gives a recursion for $p(F_n)$ that follows from Proposition 4.4 and repeated application of deletion and contraction to the left-most edge of F_n. Rooted graphs which arise during this process are either the direct sums of rooted paths with smaller rooted fans, or smaller rooted fans with paths attached to the leftmost (non-root) vertex. Determining the characteristic polynomial of these rooted graphs follows from Propositions 2.5, 4.2 and 4.5. We omit the proof.

Proposition 5.1. *Let* $n \geqslant 2$. *Then*

$$p(F_n) = (-1)^{n+1}(\lambda-1) + (-1)^n(\lambda-1)p(F_1) + (-1)^{n-1}(\lambda-1)p(F_2) + \cdots$$
$$+(-1)(\lambda-1)p(F_{n-2}) - p(F_{n-1}).$$

We can use this formula to get a simple recursion:

Corollary 5.2. $p(F_n) = -2p(F_{n-1}) - \lambda p(F_{n-2})$.

Proof. Note that using the formula of Proposition 5.1, the terms of $p(F_n) + p(F_{n-1})$ telescope, so $p(F_n) + p(F_{n-1}) = -\lambda p(F_{n-2}) - p(F_{n-1})$. □

Proposition 5.3. (1) *The constant term of* $p(F_n)$ is $(-1)^n 2^{n-1}$.

(2) *The degree of* $p(F_n)$ *is* $\lfloor (n+1)/2 \rfloor$.

(3) *Let* a_n *be the coefficient of the highest power of* λ. *Then*

(a) $a_{2k} = (-1)^k 2k$,

(b) $a_{2k+1} = (-1)^k$.

Proof. (1) There are 2^{n-1} ways to acyclically orient the edges of F_n so that $*$ is the unique source (since the edges adjacent to $*$ must be oriented away from $*$ and the remaining $n-1$ edges can be oriented arbitrarily). The result follows from Theorem 4.8.

(2) If n is even, then there is a minimum tree cover with $n/2$ edges. Thus, by Theorem 4.6, the degree of $p(F_n)$ is $n - n/2 = \lfloor (n+1)/2 \rfloor$. If n is odd, then the minimum tree cover has $(n-1)/2$ edges. The result now follows from the same proposition.

(3) We use induction on n, with 2 base cases: $p(F_1) = \lambda - 1$, so $a_1 = 1$, and $p(F_2) = -2(\lambda - 1)$, so $a_2 = -2$.

(a) Assume $k \geqslant 2$. Part 2 above yields $\deg(p(F_{2k})) = \deg(p(F_{2k-1})) = \deg(p(F_{2k-2})) + 1$. Since $p(F_{2k}) = -2p(F_{2k-1}) - \lambda p(F_{2k-2})$ by Corollary 5.2, $a_{2k} = -2a_{2k-1} - a_{2k-2} = (-2)(-1)^{k-1} - (-1)^{k-1}(2k-2) = (-1)^k 2k$.

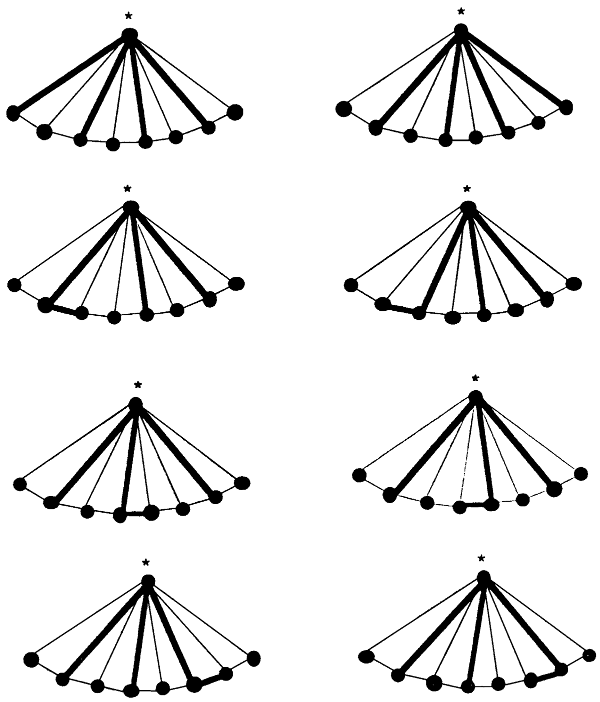

Fig. 2. The 8 minimum tree covers for F_8.

(b) Assume $k \geqslant 1$. In this case, $\deg(p(F_{2k+1})) = \deg(p(F_{2k})) + 1 = \deg(p(F_{2k-1})) + 1$. Again, using Corollary 5.2, $a_{2k+1} = -a_{2k-1} = (-1)^k$. □

Remark. Parts 2 and 3 above, combined with Corollary 4.7, tells us how many minimum tree covers T of F_n there are. When n is even, there are precisely n such subtrees, and when n is odd, there is only one. In the case when n is odd, this is obvious: if edges $e_1, \ldots, e_n$ are the (ordered) edges adjacent to the root, then the edges of T are $e_2, e_4, \ldots, e_{n-1}$. When n is even, however, the result is less obvious. For example, when $n = 8$, we find 8 minimum tree covers (see Fig. 2), so by the proposition, these must be all of them.

Recall that the polynomial $x^n - 1 = \prod_{d|n} g_d(x)$, where $g_d(x)$ is the dth *cyclotomic polynomial*. $g_d(x)$ is a monic polynomial of degree $\phi(d)$ (the Euler-ϕ function) which is irreducible over the rationals. A homogeneous version of the cyclotomic polynomial is given by

$$\alpha^n - \beta^n = \prod_{d|n} \beta^{\phi(n)} g_d(\alpha/\beta).$$

Elementary properties of these polynomials can be found in most abstract algebra texts. A standard reference is [11].

The following formulæ for $p(F_n)$ indicate the close connection between $p(F_n)$ and the cyclotomic polynomials.

Proposition 5.4. *Let F_n be the fan with $n+1$ vertices.*

1. $p(F_n;\lambda)=(-1)^n(\sqrt{1-\lambda}/2)[(1+\sqrt{1-\lambda})^n-(1-\sqrt{1-\lambda})^n]$
2. *If* $u=(1+\sqrt{1-\lambda})/(1-\sqrt{1-\lambda})$, *then* $p(F_n)=(-1)^n 2^{n-1}(u-1)(u^n-1)/(u+1)^{n+1}$

Proof. (1) We need to solve the recurrence relation of Corollary 5.2. Let $f(z)=\sum_{n\geqslant 1} p(F_n;\lambda)z^n$ be the ordinary generating function associated with the sequence of polynomials $\{p(F_n;\lambda)\}$. Using standard techniques, we get

$$f(z)=\frac{(\lambda-1)z}{1+2z+\lambda z^2}=\frac{\sqrt{1-\lambda}}{2}\left(\frac{1}{1-\alpha z}-\frac{1}{1-\beta z}\right),$$

where $\alpha=-1-\sqrt{1-\lambda}$ and $\beta=-1+\sqrt{1-\lambda}$. The result now follows immediately.

(2) This follows from 1 by using the indicated substitution. □

Definition. Let $\alpha=-1-\sqrt{1-\lambda}$, $\beta=-1+\sqrt{1-\lambda}$, and let $g_n(x)$ be the nth cyclotomic polynomial.

- For $n=1$, define $I_1(\lambda)=\lambda-1$.
- For $n\geqslant 2$, define $I_n(\lambda)=\beta^{\phi(n)}g_n(\alpha/\beta)$.

We can rewrite Formula 1 of Proposition 5.4 using these $I_n(\lambda)$.

Lemma 5.5. *For all $n\geqslant 1$, $p(F_n;\lambda)=\prod_{d|n} I_d(\lambda)$.*

Proof. Rewrite the first formula of Proposition 5.4 as follows:

$$\begin{aligned} p(F_n;\lambda) &= \frac{\sqrt{1-\lambda}}{2}(\alpha^n-\beta^n) \\ &= \frac{\sqrt{1-\lambda}}{2}(\alpha-\beta)\prod_{d|n,1<d} I_d(\lambda). \end{aligned}$$

But $(\sqrt{(1-\lambda)}/2)(\alpha-\beta)=\lambda-1=I_1(\lambda)$, so we are done. □

Lemma 5.6. *For all $n\geqslant 1, I_n(\lambda)$ is irreducible in the polynomial ring $\boldsymbol{Z}[\lambda]$.*

Proof. We first show $I_n(\lambda)$ is a polynomial in λ with integer coefficients. We use induction on n. The result is trivial for $n=1$. Now use Lemma 5.5 to write

$$p(F_n;\lambda)=\left(\prod_{d|n,d<n} I_d(\lambda)\right) I_n(\lambda)=r(\lambda)I_n(\lambda),$$

where $r(\lambda)=\prod_{d|n,d<n} I_d$. By induction, $I_d(\lambda)$ is a polynomial in λ for all $1\leqslant d<n$. Thus $r(\lambda)$ is a polynomial in λ with integer coefficients. Since $p(F_n;\lambda)$ is a polynomial

Table 1

n	$I_n(\lambda)$	$p(F_n;\lambda)$
1	$\lambda-1$	$\lambda-1$
2	-2	$-2(\lambda-1)$
3	$-\lambda+4$	$-(\lambda-1)(\lambda-4)$
4	$-2(\lambda-2)$	$4(\lambda-1)(\lambda-2)$
5	$\lambda^2-12\lambda+16$	$(\lambda-1)(\lambda^2-12\lambda+16)$
6	$-3\lambda+4$	$-2(\lambda-1)(\lambda-4)(3\lambda-4)$
7	$-\lambda^3+24\lambda^2-80\lambda+64$	$-(\lambda-1)(\lambda^3-24\lambda^2+80\lambda-64)$
8	$2(\lambda^2-8\lambda+8)$	$8(\lambda-1)(\lambda-2)(\lambda^2-8\lambda+8)$
9	$-\lambda^3+36\lambda^2-96\lambda+64$	$(\lambda-1)(\lambda-4)(\lambda^3-36\lambda^2+96\lambda-64)$
10	$5\lambda^2-20\lambda+16$	$-2(\lambda-1)(\lambda^2-12\lambda+16)\cdot$ $(5\lambda^2-20\lambda+16)$
11	$-\lambda^5+60\lambda^4-560\lambda^3$ $+1792\lambda^2-2304\lambda+1024$	$-(\lambda-1)(\lambda^5-60\lambda^4+560\lambda^3$ $-1792\lambda^2+2304\lambda-1024)$
12	$\lambda^2-16\lambda+16$	$4(\lambda-1)(\lambda-2)(\lambda-4)(3\lambda-4)\cdot$ $(\lambda^2-16\lambda+16)$

in λ with integer coefficients, this immediately gives $I_n(\lambda)$ as a polynomial in λ with integer coefficients.

It remains to show that $I_n(\lambda)$ is irreducible. Again, the result is trivial for $n=1$. For $n>1$, $I_n(\lambda)=\beta^{\phi(n)}g_n(\alpha/\beta)$ is irreducible in the polynomial ring $\boldsymbol{Z}[\alpha,\beta]$. (This follows immediately from the irreducibility of the cyclotomic polynomial $g_n(x)$.) Furthermore, $\lambda=-\alpha^2-\alpha$ implies any non-trivial factorization $I_n(\lambda)=s(\lambda)t(\lambda)$ in the ring $\boldsymbol{Z}[\lambda]$ immediately gives a non-trivial factorization in $\boldsymbol{Z}[\alpha]\subseteq\boldsymbol{Z}[\alpha,\beta]$, contradicting the irreducibility over $\boldsymbol{Z}[\alpha,\beta]$. This completes the proof. □

We summarize the factorization information about $p(F_n,\lambda)$ in the following theorem and corollary. The proofs follow from Lemmas 5.5 and 5.6.

Theorem 5.7. *Let $p(F_n)$ be the characteristic polynomial of a rooted fan on $n+1$ vertices. Then $p(F_n)=\prod_{d|n}I_d(\lambda)$ is a complete factorization of $p(F_n)$ into irreducible polynomials over $\boldsymbol{Z}[\lambda]$ (equivalently $\boldsymbol{Q}[\lambda]$).*

Corollary 5.8. (1) *$p(F_m)|p(F_n)$ if and only if $m|n$.*

(2) *$p(F_q)/(\lambda-1)$ is irreducible if and only if q is prime.*

(3) *Let $n>1$ be an integer, and let $d_1,d_2,\ldots,d_m$ be the list of proper divisors of n. Then $I_n(\lambda)=p(F_n)/\mathrm{lcm}(p(F_{d_1}),p(F_{d_2}),\ldots,p(F_{d_m}))$.*

We conclude by exhibiting in Table 1 the polynomials $I_n(\lambda)$ and $p(F_n;\lambda)$ for $1\leqslant n\leqslant 12$.

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