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Convexity and the Beta Invariant*

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Abstract. We apply a generalization of Crapo's β invariant to finite subsets of $\Re^n$. For a finite subset C of the plane, we prove $\beta(C) = |\mathrm{int}(C)|$, where $|\mathrm{int}(C)|$ is the number of interior points of C, i.e., the number of points of C which are not on the boundary of the convex hull of C. This gives the following formula: $\sum_{K \,\text{free}} (-1)^{|K|-1}|K| = |\mathrm{int}(C)|$.

1. Introduction

We are concerned with applying a greedoid version of Crapo's β invariant to finite subsets of $\Re^n$. The main result (Theorem 4.1) shows that $\beta(S)$ counts the number of interior points of S when S is a finite subset of the plane. Our approach considers the combinatorial structure of the finite point set S which arises from the intersection of S with convex sets in $\Re^n$. This associates a meet-distributive lattice with S which is called the convex set lattice.

Finite subsets of $\Re^n$ are classical examples in abstract convexity theory, a theory which is dual (via complementation) to antimatroid theory. Antimatroids form an important class of greedoids, which have been studied in connection with convexity and algorithm design. In fact, antimatroids have been rediscovered several times, having been introduced by Dilworth in the 1940s. See [10] and pages 343–344 of [1] for short and interesting accounts of the development of antimatroids. More information about greedoids and antimatroids can be found in [1] or [9], for example.

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Convexity is a very well-studied and important area of pure and applied mathematics. Several important invariants associated with finite point sets in $\Re^n$ have been considered before. These include the Helly number, the Carathéodory number, the Radon number, and the Erdös–Skerzes number. See [4] or Section 3.4 of [9] for more information concerning combinatorial invariants in convexity.

The β invariant is a well-studied invariant in its own right. Crapo defined the β invariant of a matroid in [3]. If M is a matroid, then $\beta(M)$ is a nonnegative integer which gives information about whether M is connected and whether M is the matroid of a series–parallel network. In particular, $\beta(M) = 0$ iff M is disconnected (or M consists of a single loop) [3] and $\beta(M) = 1$ iff M is the matroid of a series–parallel network (or M consists of a single isthmus) [2]. A standard reference for many of the basic properties of $\beta(M)$ is [11].

The generalization of β from matroids to greedoids appears in [5]. In this generalization, $\beta(G)$ can be any (possibly negative) integer. As in the matroid case, $\beta(G)$ is defined using either the two-variable greedoid Tutte polynomial of [6], or, more directly, the one-variable characteristic polynomial $p(G; \lambda)$ of [7]. We work directly with $p(G; \lambda)$ here.

In Section 2 we introduce the characteristic polynomial and the β invariant for finite point sets. We define $\beta(G)$ in terms of the characteristic polynomial and give two alternate formulations, including an important deletion–contraction recursion (Proposition 2.3). We also interpret deletion and contraction (which are greedoid operations) for finite subsets of $\Re^n$.

In Section 3 we give some straightforward consequences of the definitions and also prove an elementary but crucial recursion (Lemma 3.4) which generalizes the standard deletion–contraction recursion for $\beta(S)$. This lemma is used to form an inductive argument in proving the main theorem.

Section 4 gives the main theorem of this paper, a geometric interpretation for $\beta(S)$ when S is a finite set of points in the plane. In particular, we show that $\beta(S) = |\mathrm{int}(S)|$, where $\mathrm{int}(S)$ is the collection of interior points of S.

It is interesting to note that this theorem is false for other examples in abstract convexity. For example, when T is a tree, we find $\beta(T) = 2 - |V|$ (using the usual geodesic closure of vertices in the tree). When G is a chordal graph, then $\beta(G) = 1 - b(G)$, where $b(G)$ is the number of 2-connected blocks of G [5]. There is one other family in abstract convexity where the theorem is true: when T is a tree, using geodesic closure on *edges*, we get $\beta(T) = -\mathrm{int}(T)$, where $\mathrm{int}(T)$ is the number of interior edges of T, i.e., those edges of T which are not leaves [7].

Throughout this paper we assume no special familiarity with greedoids or antimatroids. Although these structures form the background for our approach, we interpret our definitions and results solely in terms of convexity.

2. Definitions and Fundamental Properties

We begin with a few familiar definitions from convexity. Let S be a finite subset of $\Re^n$. For each $A \subseteq S$, let $\tau(A) = S \cap \mathrm{conv}(A)$, where $\mathrm{conv}(A)$ is the convex hull of A. $\tau(A)$ is called the *convex closure* of A. Call a subset K of S *convex* if $\tau(K) = K$. Note that

S and $\emptyset$ are always convex. A convex set K is *free* if every subset of K is also convex. The convex hull of S forms a d-dimensional polytope, where $d \leq n$. The *boundary* of S, denoted $B(S)$, is the collection of points of S which are on the $(d-1)$-dimensional facets of this polytope. We say $x \in S$ is an *extreme* point of S if x is a vertex of this polytope. The *interior* of S, denoted $\mathrm{int}(S)$, is the complement of the boundary: $\mathrm{int}(S) = S \backslash B(S)$. As usual, $|S|$ denotes the number of points in S.

We now define the characteristic polynomial of a finite point set S.

Definition 2.1. Let S be a finite subset of $\Re^n$ with free convex sets $\mathcal{C}$. Then the *characteristic polynomial* $p(S; \lambda)$ is defined as follows:

$$p(S; \lambda) = (-1)^{|S|} \sum_{K \in \mathcal{C}} (-1)^{|K|} \lambda^{|K|}.$$

This definition appears as Proposition 7 in [7]. This definition of $p(S; \lambda)$ is motivated by the chromatic polynomial of a graph and its generalization to the characteristic polynomial of a matroid. See [7] and [11] for more background.

We now define $\beta(S)$ for $S \subseteq \Re^n$.

Definition 2.2. Let S be a finite nonempty subset of $\Re^n$ with free convex sets $\mathcal{C}$. Then $\beta(S)$ is defined as follows:

$$\beta(S) = \sum_{K \in \mathcal{C}} (-1)^{|K|-1} |K|.$$

Thus,

$$\beta(S) = (-1)^{|S|-1} p'(S; 1). \tag{2.1}$$

The next proposition (Proposition 4.2 of [5]) gives $\beta(S)$ in terms of the convex closure operator $\tau(A)$.

Proposition 2.1. *Let S be a finite subset of $\Re^n$. Then*

$$\beta(S) = \sum_{A \subseteq S} (-1)^{|A|-1} |\tau(A)|.$$

Of fundamental importance for the proofs in this paper is the recursion the characteristic polynomial satisfies. We first interpret greedoid deletion and contraction in S. We do so by specifying the convex sets in $S - x$ and S/x, where x is an extreme point of S.

Definition 2.3. Let S be a finite subset of $\Re^n$, let x be an extreme point in S, and let $C \subseteq S \backslash \{x\}$.

1. Deletion: C is convex in $S - x$ iff $C \cup \{x\}$ is convex in S.
2. Contraction: C is convex in S/x iff C is convex in S.

Deletion and contraction in greedoids are usually interpreted for feasible sets. Since we are using convex sets as our fundamental building blocks, deletion and contraction

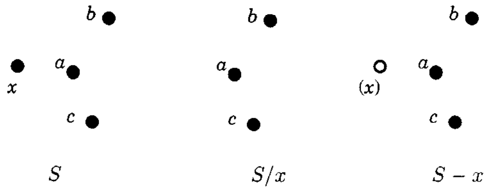


Fig. 1. Deletion and contraction in a point configuration.

are essentially reversed (in spirit) from the usual interpretations. This follows because the convex sets are simply the complements of the feasible sets.

Thus, the convex subsets of S/x are precisely the convex subsets of S which do not include the point x. We can realize this structure by simply erasing the extreme point x from the configuration S, as in Fig. 1. Unfortuately, deletion is more problematic. For deletion, we need a *pointed* version of S, so that convex subsets of $S-x$ are in one-to-one correspondence with the convex subsets of S which contain the distinguished point x. This means that the convex subsets of $S-x$ can no longer be realized by an ordinary set of points in $\Re^n$ in general. We visualize the convex sets in $S-x$ by making x a "hollow" point, as in Fig. 1. In this figure, $\{a, c\}$ is convex in $S-x$, while $\{b, c\}$ is not, since $\{a, c, x\}$ is convex in S, but $\{b, c, x\}$ is not.

Technically, we compute $p(S-x; \lambda)$ and $\beta(S-x)$ in a larger category, the category of configurations which contain both ordinary and hollow points. If K is a free convex set in a configuration T' in that larger category, then K is still a free convex in the configuration T obtained by replacing the hollow points with ordinary points. There is no ambiguity in the order in which hollow points are created, because deletion will never create new extreme points.

We define what we mean by $p(S; \lambda)$ and $\beta(S)$ in the cases where all the extreme points of S are hollow; we then give a recursion which will allow computation of $p(S; \lambda)$ and $\beta(S)$ in all cases. Finally, we demonstrate that the definition and recursion are consistent with our original definitions of $p(S; \lambda)$ and $\beta(S)$.

Definition 2.4.

1. If S consists of $k \geq 0$ hollow points, then $p(S; \lambda) = 1$ and $\beta(S) = -k$.
2. If S consists of hollow points with one or more ordinary points in the convex hull of those hollow points, then $p(S; \lambda) = 0$ and $\beta(S) = 0$.

Notice that if S consists of k hollow points alone, then those points must form a free convex subset of the original configuration. This is because each point must be extreme at the time it is deleted; and once a point becomes extreme, it remains extreme. Also notice that if S consists of hollow points with one or more ordinary points in the convex hull of those hollow points, then the ordinary points are greedoid loops. A point x in a greedoid G is a greedoid loop if it is in no feasible set of G, so x is in every convex set. Greedoid loops will not appear in ordinary point configurations; they can only occur in configurations with hollow points. Both $p(S; \lambda) = 0$ and $\beta(s) = 0$ if S has greedoid loops; this follows from the definition of $p(S; \lambda)$ in [7].

The proof of the next result follows from our definitions of deletion and contraction and Proposition 3 of [7].

Proposition 2.2. *Let x be an extreme point of a set S. Then*

$$p(S;\lambda) = \lambda p(S-x;\lambda) - p(S/x;\lambda).$$

The next proposition, which appears as Proposition 2.2 in [5] (in terms of greedoids), gives the recursion for $\beta(S)$ in terms of deletion and contraction.

Proposition 2.3. *Let x be an extreme point of S. Then*

$$\beta(S) = \beta(S/x) - \beta(S-x).$$

Proof. Assume $|S| > 1$. We differentiate the recursive formula in Proposition 2.2 to obtain

$$\beta(S) = (-1)^{|S|-1} p'(S;\lambda) = (-1)^{|S|-1}\left[p(S-x;\lambda) + \lambda p'(S-x;\lambda) - p'(S/x;\lambda)\right].$$

Proposition 5 of [7] shows that $p(S;1) = 0$ for $S \neq \emptyset$, so we multiply through by $(-1)^{|S|-1}$, set $\lambda = 1$, and use (2.1) for $\beta(S)$ to obtain $\beta(S) = (-1)(\beta(S-x) - \beta(S/x))$, as desired.

If $S = \{x\}$, then $\beta(S) = 1$. $S/x = \emptyset$ and $S - x$ consists of a single hollow point. Thus, the recursion gives $\beta(S) = \beta(S/x) - \beta(S-x) = 0 - (-1) = 1$, as desired. □

Definition 2.4 and the recursive procedure are consistent with Definition 2.2. We demonstrate the correspondence in the following example.

Example 2.1. We give a simple example of how to compute $p(S;\lambda)$ or $\beta(S)$ via Propositions 2.2 and 2.3 and Definition 2.4. Let S be the configuration consisting of three collinear points at the top of Fig. 2. Repeated deletion and contraction yields the *computation tree* of smaller configurations of Fig. 2. Computation trees are introduced and studied in [8]. In the figure we use the convention that the left child of a configuration is obtained by contraction and the right child is obtained by deletion.

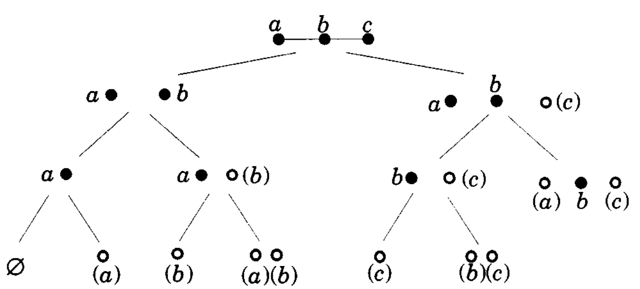


Fig. 2. A complete computation tree.

The configurations appearing at the terminal leaves of this tree consist solely of hollow points or configurations of hollow points with nonhollow points in their convex hull. The former configurations are all the free convex sets of S. The latter configurations were convex sets which were not free in the original configuration; they do not contribute to $p(S;\lambda)$ or $\beta(S)$ by Definition 2.4(2). To show that any free convex set will appear at the bottom of the computation tree, let K be a free convex set. At each splitting of the binary tree, if the point being deleted and contracted is in K, follow the deletion side, otherwise follow the contraction side. Since free convex sets are those where every point is extreme, the points not in K will be contracted and thus removed, while the points in K are deleted, become hollow, and are left at the bottom of the computation tree. This allows us to "read off" the terms in the expansions for $p(S;\lambda)$ or $\beta(S)$ given by Definitions 2.1 and 2.2 from the tree. By Propositions 2.2 and 2.3 and Definition 2.4, we get $p(S;\lambda) = -2\lambda^2 + 3\lambda - 1$ and $\beta(S) = 2(-2) + 3(1) + 0 = -1$. (Note that the powers of λ arise from the factor of λ appearing as a coefficient of $p(S - x)$ in the recursion for $p(S)$.)

For $\beta(S)$, we remark that a configuration of k hollow points appearing at the bottom of the computation tree yields a summand of $(-1)^{k-1}(-k)$ by Proposition 2.3 and Definition 2.4(1). However, this is precisely the summand a free convex set contributes to $\beta(S)$ by Definition 2.2.

Thus, the computation tree gives us an expansion for both $p(S)$ and $\beta(S)$. This example shows how it is possible in general to see that the recursive procedures of Propositions 2.2 and 2.3 and Definition 2.4 are consistent with our original Definitions 2.1 and 2.2 of the characteristic polynomial and the β invariant.

We finally note that we now have three different ways to compute $\beta(S)$: via Definition 2.2, the expansion of Proposition 2.1, and the recursive procedure outlined above. A more complicated version of the recursion which holds in the plane is given in Lemma 3.4. This recursion will be the key step in proving the main theorem.

3. Some Preliminary Results for $p(S;\lambda)$ and $\beta(S)$

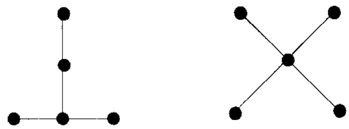
Although the characteristic polynomial determines the number of free convex subsets of size k for all k, it does not uniquely determine the combinatorial structure of S. In fact, the following example shows that the number of extreme points in S is not even determined by the polynomial.

Example 3.1. Let S_1 and S_2 be the two configurations of Fig. 3. Then

$$p(S_1;\lambda) = p(S_2;\lambda) = 4\lambda^3 - 8\lambda^2 + 5\lambda - 1.$$

Note that S_1 has three extreme points while S_2 has four. Note that $\beta(S_1) = \beta(S_2) = 1$; each configuration has one interior point (see Theorem 4.1 below).

Our next result simply computes $p(S;\lambda)$ for some standard configurations which we will need later. The proofs are straightforward.



S_1 S_2

Fig. 3. Configurations with the same characteristic polynomial.

Proposition 3.1. *Let S be a finite set of points in* $\Re^n$.

1. *If all points in S are extreme, then*

$$p(S;\lambda) = (-1)^{|S|}(1-\lambda)^{|S|}.$$

2. *If the finite point set S consists of k points on a line, then*

$$p(S;\lambda) = (-1)^k(1-\lambda)(1-(k-1)\lambda).$$

3. *If the finite point set S consists of* $k-1$ *points on a line and a single point not on that line, then*

$$p(S;\lambda) = (-1)^k(1-\lambda)^2(1-(k-2)\lambda).$$

The following result gives values for some of the coefficients of the characteristic polynomial. The proof follows from considering the free convex sets of the corresponding sizes and applying Definition 2.1.

Proposition 3.2. *Let S be a finite set of k points in* $\Re^n$ *and let*

$$p(\lambda) = (-1)^k(1 - a_1\lambda + a_2\lambda^2 - \cdots + (-1)^{k-1}a_{k-1}\lambda^{k-1} + (-1)^k a_k\lambda^k),$$

where a_i *is the number of free convex sets of size k. Then*

1. $a_1 = k$,
2. *if* l_m *is the number of m-point lines, then* $a_2 = \sum_m (m-1)l_m$,
3. $$a_k = \begin{cases} 1 & \textit{if all points in S are extreme,} \\ 0 & \textit{otherwise.} \end{cases}$$

We also need the following facts about $\beta(S)$. The proofs follow from Proposition 3.1 and the definitions.

Proposition 3.3. *Let S be a finite set of points in* $\Re^n$.

1. *If S is a set of k points on a line, then* $\beta(S) = 2-k$.
2. *If all points in S are extreme, then* $\beta(S) = 0$; *in particular, if S is the set of* $n+1$ *vertices of a simplex in* $\Re^n$, *then* $\beta(S) = 0$.
3. *If S consists of the* $n+1$ *vertices of a simplex in* $\Re^n$ *together with the barycenter of this simplex, then* $\beta(S) = (-1)^n$.

Finally, we have the following lemma, which is of central importance in proving Theorem 4.1. We note that contraction of several extreme points is a commutative operation, so we write S/xy for $(S/x)/y = (S/y)/x$.

Lemma 3.4. *Let S be a finite set of points in the plane, not all on a line, with at least one interior point. Then there must exist three extreme points x, y, and z with some other point w in their convex closure. In this case,*

$$\beta(S) = \beta(S/x) + \beta(S/y) + \beta(S/z) - \beta(S/xy) - \beta(S/xz) - \beta(S/yz) + \beta(S/xyz).$$

Proof. The existence of x, y, and z follows by triangulating the convex hull of S. If w is an interior point of S, then w lies in one of the triangles,

We prove the formula using repeated applications of the recursion $\beta(S - x) = \beta(S/x) - \beta(S)$, which is obtained from Proposition 2.3. We also note that, because x, y, and z are all extreme points, deletions and contractions of several points may be done in any order. Then

$$\begin{aligned}
\beta(S) &= \beta(S/x) - \beta(S - x) \\
&= \beta(S/x) - [\beta(S - x/y) - \beta(S - x - y)] \\
&= \beta(S/x) - [\beta(S/xy) - \beta(S/y)] + [\beta(S - x - y/z) - \beta(S - x - y - z)] \\
&= \beta(S/x) + \beta(S/y) - \beta(S/xy) + [\beta(S/yz - x) - \beta(S/z - x)] \\
&\quad - \beta(S - x - y - z) \\
&= \beta(S/x) + \beta(S/y) - \beta(S/xy) + [\beta(S/xyz) - \beta(S/yz)] \\
&\quad - [\beta(S/xz) - \beta(S/z)] - \beta(S - x - y - z) \\
&= \beta(S/x) + \beta(S/y) + \beta(S/z) - \beta(S/xy) - [\beta(S/xz) - \beta(S/yz)] \\
&\quad + \beta(S/xyz) - \beta(S - x - y - z).
\end{aligned}$$

The result now follows from the fact that $\beta(S - x - y - z) = 0$, from Definition 2.4(2). □

This result generalizes to higher-dimensional configurations, but we will not need these generalizations here.

4. The Main Theorem

In this section we state and prove our main result about $\beta(S)$ when S is a subset of the plane.

Theorem 4.1. *Let S be a finite subset of the plane which does not lie on a line. Then*

$$\beta(S) = |\mathrm{int}(S)|.$$

The proof of Theorem 4.1, which we give below, is structured as follows: We use the formula from Lemma 3.4 and induction to rewrite $\beta(S)$ as a sum (or difference) of $|\mathrm{int}(T)|$ for the seven smaller configurations T appearing on the right-hand side of the formula. We first treat the degenerate cases in which the induction hypothesis cannot be applied, i.e., when the points of some subconfiguration T all lie on a line.

We should note here that our notion of interior is simply the topological *relative* interior. For example, if S consists of k points on a line, then S will have $k-2$ interior points; in this case, $\beta(S) = (-1)(k-2)$ by Proposition 3.3(1).

We then assume each subconfiguration T appearing in the formula from Lemma 3.4 is two-dimensional, so that the inductive hypothesis allows us to replace $\beta(T)$ with $|\mathrm{int}(T)|$ for each T. We then show that every interior point p of S is counted precisely once on the right-hand side of the formula by a careful case-by-case analysis of which subconfigurations T do and do not contain p as an interior point. This has an inclusion–exclusion flavor, which is not surprising in light of the formula of the lemma.

Before proving the theorem, we prove a special case as a lemma.

Lemma 4.1. *Let S be a finite subset of the plane, with no interior points. Then* $\beta(S) = 0$.

Proof. If every point of S is an extreme point, then $\beta(S) = 0$ by Proposition 3.3(2). In particular, when the convex hull of S is a line segment, then S consists of two points (since S has no interior points), so $\beta(S) = 0$.

We now assume the convex hull of S is two-dimensional and proceed by induction on $n = |S|$. The smallest configuration S with a two-dimensional convex hull containing nonextreme, noninterior points is the configuration consisting of a three collinear points with one point off the line. It is easy to check that $\beta(S) = 0$ for this configuration.

We now assume $n > 4$ (and that S has a two-dimensional convex hull which contains no interior points). We may further assume that S contains nonextreme, noninterior points (or else S contains only extreme points). Then there are extreme points x and y in S so that the line segment $\overline{xy}$ contains some other point w of S in its (relative) interior.

As in the proof of Lemma 3.4, we get a reduction formula for $\beta(S)$ solely in terms of contraction.

$$\beta(S) = \beta(S/x) + \beta(S/y) - \beta(S/xy).$$

Since $n > 4$, the smaller configurations S/x, S/y, and S/xy all have two-dimensional convex hulls (and, of course, these configurations contain no interior points). Then $\beta(S/x) = \beta(S/y) = \beta(S/xy) = 0$ by induction, so we are done. □

We are now ready to prove Theorem 4.1.

Proof. We may assume S contains interior points and the convex hull of S is two-dimensional. In order to apply induction to the configuration S, we use the formula of Lemma 3.4.

Let $V \subseteq \{x, y, z\}$. When all of the points in the contracted configuration S/V lie on a line, the induction hypothesis does not apply to $\beta(S/V)$. Therefore, we must consider several degenerate configurations separately in the inductive proof, applying Lemma 4.1

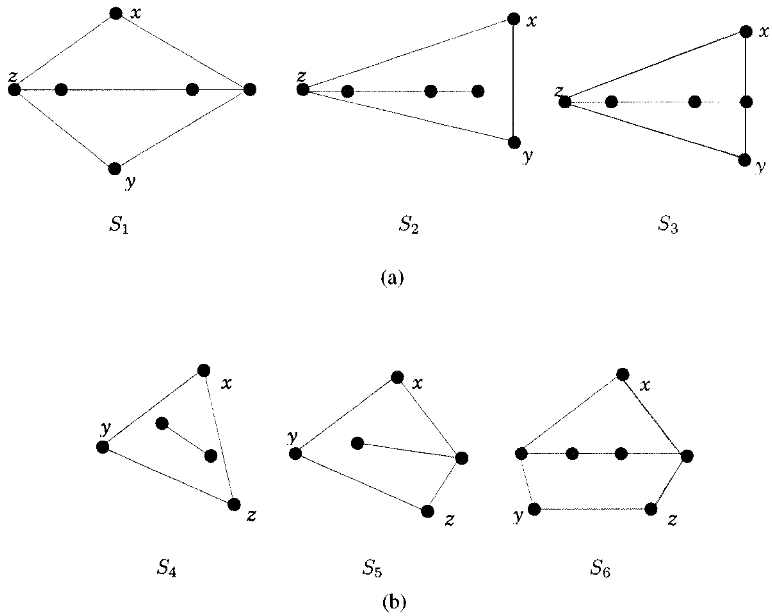


Fig. 4. The three types of configurations in which (a) S/xy is linear and (b) S/xyz is linear.

and Proposition 3.3(1) whenever necessary, i.e., whenever the contraction of x, y, and z reduces the dimension of the convex hull of S. For those degenerate cases in which there are no interior points in S, $\beta(S) = 0$ by Lemma 4.1.

For each of the degenerate configurations S_i considered, we let $k = |\text{int}(S_i)|$. For each of the configurations of Fig. 4(a) and (b), we use the formula from Lemma 3.4. The results are summarized in Table 1, where we compute $\beta(S_i)$ for all the degenerate configurations. In each case, $\beta(S_i) = k$ by the lemma, as required.

We are now ready to prove the general case by induction on the number of points in S. The base case consists of all configurations of fewer than six points, which are included among the degenerate cases considered above.

Table 1. Computing $\beta(S)$ for six degenerate configurations.

S	$\beta(S/x)$	$\beta(S/y)$	$\beta(S/z)$	$\beta(S/xy)$	$\beta(S/xz)$	$\beta(S/yz)$	$\beta(S/xyz)$
S_1	$0^{(1)}$	$0^{(1)}$	$k-1^{(3)}$	$-k^{(2)}$	$0^{(1)}$	$0^{(1)}$	$1-k^{(2)}$
S_2	$0^{(1)}$	$0^{(1)}$	$k-1^{(3)}$	$1-k^{(2)}$	$0^{(1)}$	$0^{(1)}$	$2-k^{(2)}$
S_3	$0^{(1)}$	$0^{(1)}$	$k-1^{(3)}$	$-k^{(2)}$	$0^{(1)}$	$0^{(1)}$	$1-k^{(2)}$
S_4	$0^{(1)}$	$k-1^{(3)}$	$k-1^{(3)}$	$0^{(1)}$	$0^{(1)}$	$0^{(1)}$	$2-k^{(2)}$
S_5	$0^{(1)}$	$k-1^{(3)}$	$k^{(3)}$	$0^{(1)}$	$0^{(1)}$	$0^{(1)}$	$1-k^{(2)}$
S_6	$0^{(1)}$	$k^{(3)}$	$k^{(3)}$	$0^{(1)}$	$0^{(1)}$	$0^{(1)}$	$-k^{(2)}$

$^{(1)}$ By Lemma 4.1.
$^{(2)}$ By Proposition 3.1(1).
$^{(3)}$ By induction.

We now assume that S consists of n points whose convex hull is two-dimensional, where $n \geq 6$, that S has at least one point in its interior, and that all configurations T of fewer than n points (whose convex hulls are two-dimensional) satisfy $\beta(T) = |\text{int}(T)|$. Thus, Lemma 3.4 gives three extreme points x, y, and z in S whose convex closure contains a fourth point w of S. We assume further that S/xyz is two-dimensional (since the situation where it is not has been taken care of above). Then again by Lemma 3.4, we know that

$$\beta(S) = \beta(S/x) + \beta(S/y) + \beta(S/z) - \beta(S/xy) - \beta(S/xz) - \beta(S/yz) + \beta(S/xyz).$$

By induction, if V is a nonempty subset of $\{x, y, z\}$, then $\beta(S/V) = |\text{int}(S/V)|$.

We first make a general observation that proves quite useful. Let $p \in S$ and let $T \subset S$. Recall that $p \notin \text{int}(S/T)$ (where S/T is the subconfiguration of S obtained by removing (contracting) the points of T) iff $p \in B(S/T)$, the boundary of S/T. If $U \subseteq T \subseteq S$, and neither S/T nor S/U consists solely of points on a line, then $p \notin \text{int}(S/U)$ implies $p \notin \text{int}(S/T)$. This follows because if p has become a boundary point when some points are contracted, then p will remain a boundary point if more points are contracted. The contrapositive of this statement is

$$\text{if } U \subseteq T \subseteq S \text{ and } p \in \text{int}(S/T), \text{ then } p \in \text{int}(S/U). \qquad (\dagger)$$

The converse is false; see Fig. 7 and the discussion following this proof.

We now wish to show that the total contribution of $p \in S$ to the right-hand side of the recursion of Lemma 3.4 equals 0 when $p \notin \text{int}(S)$ and equals 1 when $p \in \text{int}(S)$ by adding the positive and negative contributions of p to each of the seven terms appearing in the recursion.

If $p \notin \text{int}(S)$, then $p \notin \text{int}(S/V)$ for each nonempty subset V of $\{x, y, z\}$, by (†). Thus the contribution of p to $\beta(S/V)$ is 0 for each such V, so the contribution of p to $\beta(S)$ is 0 by Lemma 3.4.

Now assume $p \in \text{int}(S)$. We consider five cases:

Case 1: $p \in \text{int}(S/xyz)$. By (†), $p \in \text{int}(S/xy)$, $p \in \text{int}(S/xz)$, $p \in \text{int}(S/yz)$, $p \in \text{int}(S/x)$, $p \in \text{int}(S/y)$, and $p \in \text{int}(S/z)$. Thus, the total contribution of p to $\beta(S)$ is $1 + 1 + 1 - 1 - 1 - 1 + 1 = 1$, as desired.

For the remaining four cases, we note that since $p \notin \text{int}(S/xyz)$, p is a boundary point of S/xyz. There are two types of configurations, which we label A and B, where this can happen; see Fig. 5. Each configuration subdivides the plane into four regions, which we label I, II, III, and IV, as shown. For our purposes, region III does not include the points on its bounding line(s), while regions II and IV do. Our goal is to determine where x, y, and z were in the original configuration.

Case 2: $p \in \text{int}(S/xy)$, $p \in \text{int}(S/xz)$, *and* $p \in \text{int}(S/yz)$. First suppose S/xyz is Type A. Then $p \in \text{int}(S/xy)$ forces $z \in$ III. Similarly, $p \in \text{int}(S/xz)$ and $p \in \text{int}(S/yz)$ force $x, y \in$ III. However, $\{x, y, z\} \subseteq$ III implies the convex closure of $\{x, y, z\}$ contains no interior point w of S, contradicting the choice of x, y, and z.

Now if S/xyz is Type B, the same argument again forces $\{x, y, z\} \subseteq$ III, again contradicting the choice of x, y, and z. Thus, this case cannot occur.

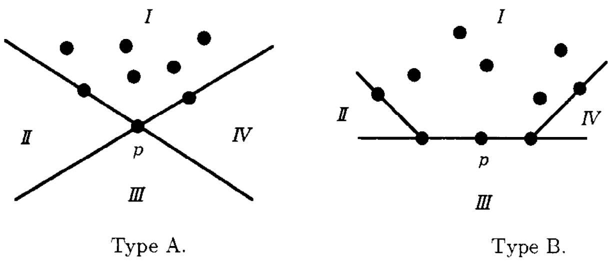


Fig. 5. The two possible configurations for S/xyz.

Case 3: $p \in \text{int}(S/xy)$, $p \in \text{int}(S/xz)$, *but* $p \notin \text{int}(S/yz)$. By (†), $p \in \text{int}(S/x)$, $p \in \text{int}(S/y)$, and $p \in \text{int}(S/z)$. Thus, the total contribution of p to $\beta(S)$ is $1 + 1 + 1 - 1 - 1 = 1$, as desired.

Case 4: $p \in \text{int}(S/xy)$, *but* $p \notin \text{int}(S/xz)$ *and* $p \notin \text{int}(S/yz)$. By (†), $p \in \text{int}(S/x)$ and $p \in \text{int}(S/y)$; we show that $p \notin \text{int}(S/z)$. For Type A or Type B, we have $z \in$ III (as in Case 2), but $p \notin \text{int}(S/xz)$ and $p \notin \text{int}(S/yz)$ imply that $x \notin$ III and $y \notin$ III.

Now suppose $p \in \text{int}(S/z)$. For Type A or B, this forces the line segment determined by the points x and y to pass through region III. For Type B, this places either x or y in region III, which is a contradiction. For Type A, we must have $x \in$ II and $y \in$ IV (or vice versa). However, then the convex closure of $\{x, y, z\}$ contains no interior point w of S, again contradicting the choice of x, y, and z.

Thus $p \notin \text{int}(S/z)$, so the total contribution of p to $\beta(S)$ is $1 + 1 - 1 = 1$, as desired.

Case 5: $p \notin \text{int}(S/xy)$, $p \notin \text{int}(S/xz)$, *and* $p \notin \text{int}(S/yz)$. Since $p \in \text{int}(S)$, we must show that p is included in exactly one of the three sets $\text{int}(S/x)$, $\text{int}(S/y)$, or $\text{int}(S/z)$. By assumption, we know none of x, y, or z is in the interior of region III for either Type A or B. $p \in \text{int}(S)$ implies the triangle determined by x, y, and z must intersect region III. This means that (at least) one of the line segments xy, xz, or yz must pass through region III. For Type B, this places one of these points in region III, which is a contradiction.

For Type A, assume xy passes through III so that xy is a bounding line segment in the convex hull of S. Then $x \in$ II and $y \in$ IV (or vice versa), and clearly $p \in \text{int}(S/z)$. Suppose $p \in \text{int}(S/x)$. Then the segment yz would also pass through region III, so x, y, and z are situated as in Fig. 6. However, then the convex closure of $\{x, y, z\}$ contains no interior point w of S, again contradicting the choice of x, y, and z. Thus

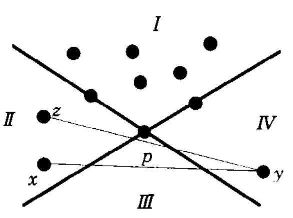


Fig. 6. The possible location of x, y, and z in Case 5.

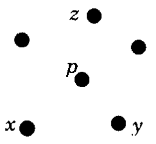


Fig. 7. $\mathrm{int}(S/xy) \neq \mathrm{int}(S/x) \cap \mathrm{int}(S/y)$.

$p \notin \mathrm{int}(S/x)$, and a similar argument shows $p \notin \mathrm{int}(S/y)$. Hence p is counted as an interior point precisely once in the recursive formula for $\beta(S)$ (Lemma 3.4), in the term $\mathrm{int}(S/z)$.

Thus, in all cases, the point p is counted precisely once on the right-hand side of our recursive formula when $p \in \mathrm{int}(S)$ and counted 0 times when $p \notin \mathrm{int}(S)$. This completes the proof. □

We note that the proof of Theorem 4.1 would be simpler if it were true that a point p is interior in S/xy if and only if p is interior in both S/x and S/y, i.e., if $\mathrm{int}(S/xy) = \mathrm{int}(S/x) \cap \mathrm{int}(S/y)$. This would allow a simple inclusion–exclusion argument to finish the entire proof (after the degenerate cases are disposed of). This is false, however, as the configuration in Fig. 7 shows: p is interior in both S/x and S/y but is not interior in S/xy. (Note that p will be interior in S/z, S/xz, and S/yz in this case, so the count from the recursive formula of Lemma 3.4 still gives 1.)

Finally, the combination of this result, part 1 of Proposition 3.3 and Proposition 3.3(3) motivates the following conjecture.

Conjecture 4.1. *Let S be a finite subset of $\Re^n$, with the convex hull of S of dimension n. Then*

$$\beta(S) = (-1)^n |\mathrm{int}(S)|.$$

The conjecture is true in one and two dimensions.

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We wish to thank Lorenzo Traldi and Tom Zaslavsky for useful discussions concerning Theorem 4.1.

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Added in proof. Paul Edelman and Victor Reiner have recently proven Conjecture 4.1 for all dimensions $n > 2$. Their proof uses methods from combinatorial topology and extends to other examples of convex geometries.