Separable extensions of noncommutative rings

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1. Introduction. Separable extensions of noncommutative rings were introduced in 1966 by K. Hirata and K. Sugano [4]. In [1] Hirata isolated a special class of separable extensions, now known as $H$-separable extensions. These have been studied extensively in a series of papers over the last fifteen years, notably by Hirata and Sugano, themselves.

A ring $A$ is an $H$-separable extension of a subring $R$ if $A \otimes_R A$ is isomorphic as $A$, $A$-bimodule to a direct summand of $A^n$, for some positive integer $n$. An $H$-separable extension is separable; i.e. the multiplication map $A \otimes_R A \to A$ splits. In the case of algebras over commutative rings, $H$-separable extensions are closely related to Azumaya algebras. In this case, $A$ is an $H$-separable extension of $R$ if $A$ is an Azumaya algebra over a (commutative) epimorphic extension of $R$.

If $A$ is a ring with subring $R$ we denote by $C$ the center of $A$ and $\Delta = A^C$, the centralizer of $R$ in $A$. Then $A$ is an $H$-separable extension of $R$ if and only if $\Delta$ is finitely generated and projective as $C$-module, and the map $\phi: A \otimes_R A \to \text{Hom}_C(\Delta, A)$ defined by $\phi(a \otimes b)(d) = adb$, for $a, b \in A$, $d \in \Delta$, is an isomorphism. There are similarly defined maps $\Delta \otimes_C A \to \text{Hom}(R^A, RA)$, $A \otimes_C \Delta \to \text{Hom}(A_R, A_R)$, and $\Delta \otimes_C \Delta \to \text{Hom}(RA_R, RA_R)$, all of which are isomorphisms when $A$ is $H$-separable over $R$. (See [12]).

In Sections 3 and 4 of this paper we generalize $H$-separability in two directions. We call $A$ a strongly separable extension of $R$ if $A \otimes_R A \cong K \oplus L$, where $\text{Hom}_{A,A}(K, A) = (0)$ and $L$ is a direct summand of $A^n$, for some positive integer $n$. $H$-separability is the case where $K = (0)$. Strong separability is equivalent to separability for algebras over a commutative ring, but not in general. We show that $A$ is strongly separable over $R$ if and only if $\Delta$ is finitely generated and projective and the map $\phi$ defined above is a split epimorphism. The three maps above which are isomorphisms in the $H$-separable case are split monomorphisms when strong separability is assumed.

If $\sigma$ is an automorphism of $A$, denote by $A_\sigma$ the $A$, $A$-bimodule which as left $A$-module is just $A$ but whose right $A$-module structure is "twisted" by $\sigma$. Then $A$ is a psuedo-Galois extension of $R$ if there is a finite set $S$ of $R$-automorphisms of $A$ such that $A \otimes_R A$ is a direct summand of $\sum_{\sigma \in S} A_\sigma^n$,
some positive integer $n$. $H$-separability is the case $S=\{1\}$. When $A$ is a Galois extension of $R$, it is pseudo-Galois, and this is the motivation for the name.

Assume $A$ is a pseudo-Galois extension of $R$ and that for all $\sigma, \tau \in \text{Aut}_R(A)$ any nonzero $A$, $A$-bimodule map from $A\sigma$ to $A\tau$ is an isomorphism. Then there is a positive integer $n$ and a finite subset $S$ of $\text{Aut}_R(A)$ containing exactly one element from each coset of the subgroup $I$ of inner automorphisms such that $A \otimes_R A \cong \sum_{\sigma} \oplus \text{Hom}_C(\Delta, A_\sigma)$, and $\text{Hom}_C(\Delta, A_\sigma)$ is isomorphic to a direct summand of $A_{\sigma}^n$, each $\sigma \in S$. Under these assumptions, $A$ is strongly separable over $R$.

In Section 2 we show that if $A$ is an $H$-separable extension of $R$ which is generated over $R$ by the centralizer $\Delta$ of $R$, and if $R$ contains the center $C$ of $A$, then $\Delta$ is an Azumaya algebra over $C$ and $A \cong \Delta \otimes_C R$. This conclusion has been obtained for $H$-separable extensions under other hypotheses by Hirata [2].

2. Assume $A$ is a separable extension of $R$ and let $M$ be a left or right $A$-module. Sugano [12] has shown that if $M$ is projective (injective) as $R$-module then it is also projective (injective) as $A$-module. An immediate consequence of this is that a separable extension of a semisimple artinian ring is also semisimple artinian. Sugano has shown further that if $A$ is flat as left or right $R$-module, then $A$ is quasi-Frobenius if $R$ is. A related result is the following.

**Proposition 2.1.** Let $A$ be a separable extension of $R$ such that $A$ is flat as left (resp. right) $R$-module. Then $A$ is left (resp. right) perfect if $R$ is.

**Proof.** Recall that a ring is left perfect if every flat left module is projective. Assume $R$ is left flat and $M$ is a flat left $A$-module. We show that $_RM$ is flat. Let $(0) \rightarrow N \rightarrow N'$ be an exact sequence of right $R$-modules. Then $(0) \rightarrow N \otimes_R A \rightarrow N' \otimes_R A$ is exact because $RA$ is flat. Thus $(0) \rightarrow N \otimes_R A \otimes_A M \rightarrow N' \otimes_R A \otimes_A M$ is exact, by the flatness of $AM$. So $(0) \rightarrow N \otimes_R M \rightarrow N' \otimes_R M$ is exact, and $RM$ is flat. Then $RM$ is projective because $R$ is perfect, and $AM$ is projective by the result mentioned above. Therefore $A$ is left perfect.

If $\Delta$ is an Azumaya algebra over its center $C$ and $R$ is a central $C$-algebra, then it is easy to see that $A = \Delta \otimes_C R$ is an $H$-separable extension of $R$. Furthermore, the centralizer of $R$ in $A$ is $\Delta$. The following Theorem is a converse to this observation.

**Theorem 2.2.** Let $A$ be an $H$-separable extension of a subring $R$ such
that $A$ is generated over $R$ by its centralizer $\Delta$ in $A$. Assume that the center $C$ of $A$ is contained in $R$. Then $\Delta$ is an Azumaya algebra over $C$, and $A \cong \Delta \otimes_C R$.

**Proof.** First, define $\phi: \Delta \otimes_C A \rightarrow A \otimes_R A$ by $\phi(d \otimes a) = d \otimes a \in A \otimes_R A$. This map is well-defined because $C \subseteq R$.

Since $A = \Delta R$, each element of $A \otimes_R A$ can be written in the form $\sum d_i \otimes b_i$ with $d_i \in \Delta$, $b_i \in A$.

Define $\phi: A \otimes_R A \rightarrow \Delta \otimes_C A$ by $\phi(\sum d_i \otimes b_i) = \sum d_i \otimes b_i \in \Delta \otimes_C A$. We need to show that $\phi$ is well-defined. Assume $\sum d_i \otimes b_i = 0$ in $A \otimes_R A$. Since $A$ is $H$-separable over $R$, $A \otimes_R A \cong \text{Hom}_C(\Delta, A)$ under the map $a \otimes b \mapsto [d \mapsto adb]$, and $\Delta \otimes_C A \cong \text{Hom}(\Delta \otimes_C R, R \otimes A)$ under the map $d \otimes b \mapsto [x \mapsto dxb]$. From $\sum d_i \otimes b_i = 0$ in $A \otimes_R A$ we have $\sum d_i \otimes b_i = 0$, for all $d \in \Delta$. Let $x \in A$, and write $x = \sum r_j e_j$, $r_j \in R$, $e_j \in \Delta$. Then $\sum d_i \otimes b_i = \sum d_i r_j e_j b_i = \sum r_j \sum d_i e_j b_i = 0$. Thus $\sum d_i \otimes b_i$ determines the zero element of $\text{Hom}(\Delta \otimes_C R, R \otimes A)$, and so $\sum d_i \otimes b_i = 0$ in $\Delta \otimes_C A$. It follows that $\phi$ is a well-defined map. Clearly, $\phi$ and $\phi$ are inverse isomorphisms, $A \otimes_R A \cong \Delta \otimes_C A$.

Since $A$ is $H$-separable over $R$, $\Delta$ is finitely generated and projective, hence flat. Thus,

$$(0) \rightarrow R \rightarrow A \text{ exact yields } (0) \rightarrow \Delta \otimes_C R \rightarrow A \otimes_R A \cong A \otimes_R A \text{ exact.}$$

So the natural map $\Delta \otimes_C R \rightarrow A \otimes_R A$ is injective. The multiplication map $f: \Delta \otimes_C R \rightarrow A$, $d \otimes r \mapsto dr$, is surjective by hypothesis. We show $f$ is also injective. Assume $\sum d_i r_i = 0$, $d_i \in \Delta$, $r_i \in R$. Then under the injective map $\Delta \otimes_C R \rightarrow A \otimes_R A$, $\sum d_i \otimes r_i \mapsto \sum d_i r_i \otimes 1 = 0$. Hence, $\sum d_i \otimes r_i = 0$ in $\Delta \otimes_C R$, and $f$ is injective. This proves $A \cong \Delta \otimes_C R$.

From $A \cong \Delta \otimes_C R$ it follows easily that $C$ is the center of $\Delta$. Also, $C$ is a direct summand of $\Delta$ (see, for example, Hirata [1], p. 112), which implies that $\Delta R_R_R$ is a direct summand of $\Delta R_R_R = \Delta R_R$.

So we can apply Prop. 4.7 of [2] to conclude that $\Delta$ is an Azumaya algebra over $C$.

3. **Strongly separable extensions.** Many results which hold for $H$-separable extensions can be extended in weakened form to a much larger class of separable extensions, which we call strongly separable.

**Definition 3.1.** $A$ is said to be strongly separable over $R$ provided $\Delta$ is finitely generated and projective as $C$-module, and the map $\phi: A \otimes_R A \rightarrow \text{Hom}_C(\Delta, A)$ is surjective and splits.

An $H$-separable extension is strongly separable, and we show now that
a strongly separable extension is separable. We will also see that for an algebra over a commutative ring, strong separability and separability are equivalent. We will present an example to show that this equivalence does not hold in general.

**Proposition 3.2.** If $A$ is strongly separable over $R$ then $A$ is separable over $R$.

**Proof.** Since $\Delta_C$ is finitely generated and projective, $C$ is a direct summand of $\Delta_C$ (see, for example, Hirata [1], p. 112). Thus the map $\phi: \text{Hom}_C(\Delta, A) \to A$, $f \mapsto f(1)$, splits as $A$, $A$-bimodule map. Let $\phi'$ be the splitting map. Also, let $\phi'$ be the splitting map for $\phi$. We have the commutative diagram

$$
\begin{array}{ccc}
A \otimes_R A & \xrightarrow{\phi} & \text{Hom}_C(\Delta, A) \\
& \searrow \mu & \downarrow \phi' \\
& & A \\
& \nearrow \psi & \\
& \phi & \\
\end{array}
$$

and it is seen that the map $\mu$ is split by $\phi' \circ \phi$. Hence $A$ is separable over $R$.

**Proposition 3.3.** If $A$ is a separable algebra over a commutative ring $R$ then $A$ is strongly separable over $R$.

**Proof.** We have $R \subseteq C \subseteq A$; hence $\Delta = A$. Since $A$ is separable over $R$ it is an Azumaya algebra over $C$. Hence $A_C$ is faithfully projective and finitely generated, and $A \otimes_C A$ is isomorphic to $\text{Hom}_C(A, A) = \text{Hom}_C(\Delta, A)$. Also, $C$ is separable over $R$; so the sequence $C \otimes_R C \to C \to 0$ is split exact. Tensoring on the left and right with $A$ over $C$, we obtain the split exact sequence $A \otimes_R A \to A \otimes_C A \to 0$. The diagram

$$
A \otimes_R A \longrightarrow A \otimes_C A \\
\text{Hom}_C(A, A) \longleftarrow
$$

is commutative. So the sequence $A \otimes_R A \to \text{Hom}_C(A, A)$ splits, and $A$ is strongly separable over $R$. This completes the proof.

The following lemma is well-known and is stated here without proof.

**Lemma 3.4.** Let $S$ and $T$ be rings; let $U$ be a right $S$-module, $V$ an $S$, $T$-bimodule, and $W$ a left $T$-module. There are canonical maps:

$$
U \otimes_S \text{Hom}_T(V, W) \longrightarrow \text{Hom}_T(\text{Hom}_S(U, V), W), \ u \otimes f \mapsto [(g \mapsto g(u)f)],
$$
and

\[ \text{Hom}_S(V, U) \otimes_T W \longrightarrow \text{Hom}_S(\text{Hom}_T(W, V), U), \quad f \otimes w \mapsto [g \mapsto f(wg)] . \]

If \( U \) is finitely generated and projective, the first map is an isomorphism; if \( W \) is finitely generated and projective, the second map is an isomorphism.

\( A \) is \( H \)-separable over \( R \) if and only if \( A \otimes_R A \) is a bimodule direct summand of \( A^n \), for some positive integer \( n \). The following result gives an analogous characterization of strong separability.

**Theorem 3.5.** Let \( R \) be a subring of a ring \( A \). Then the following conditions are equivalent:

1. \( A \) is strongly separable over \( R \).
2. There exist \( d_i \in \Delta, \sum_j a_{ij} \otimes b_{ij} \in (A \otimes_R A)^A, 1 \leq i \leq n \), such that \( d = \sum_{i,j} d_i a_{i,j} b_{ij} \) for any \( d \in \Delta \).
3. \( A \otimes_R A = K \oplus M \), where \( \text{Hom}_{A,A}(K, A) = (0) \) and \( M \) is isomorphic to an \( A \), \( A \)-bimodule direct summand of \( A^n \).

**Proof.** ((1) \( \Rightarrow \) (3)) Assume \( A \) is strongly separable over \( R \). Then there is a split exact sequence \( C^n \longrightarrow \Delta \longrightarrow (0) \) of \( C \)-modules, because \( \Delta \) is finitely generated and projective. This yields a split exact sequence \( (0) \longrightarrow \text{Hom}_C(\Delta, A) \longrightarrow \text{Hom}_C(C^n, A) \cong A^n \) of \( A \), \( A \)-bimodules. Let \( K = \ker(\phi) \). Since

\[ (0) \longrightarrow K \longrightarrow A \otimes_R A \xrightarrow{\phi} \text{Hom}_C(\Delta, A) \longrightarrow (0) \]

splits, \( K \) is a direct summand of \( A \otimes_R A \) such that \( A \otimes_R A / K \cong \text{Hom}_C(\Delta, A) \). We need to show that \( \text{Hom}_{A,A}(K, A) = (0) \).

We apply Lemma 3.4 with \( S = C, T = A \otimes_C A, U = \Delta, V = A, W = A \), noting that \( \Delta \) is finitely generated and projective as required. Then

\[ \Delta \cong \text{Hom}_{A \otimes_C A}(A, A) \cong \text{Hom}_{A \otimes_C A}(\text{Hom}_C(\Delta, A), A) \cong \text{Hom}_{A \otimes_C A}(K, A) \cong A^n. \]

But \( \Delta \otimes_C \text{Hom}_{A \otimes_C A}(A, A) \cong \Delta \otimes C \cong \Delta \). By hypothesis, \( A \otimes_R A \cong \text{Hom}_C(\Delta, A) \oplus K \). Thus we have the following sequence of isomorphisms:

\[ \Delta \cong \text{Hom}_{A \otimes_C A}(A \otimes_R A, A) \cong \text{Hom}_{A \otimes_C A}(\text{Hom}_C(\Delta, A), A) \oplus \text{Hom}_{A \otimes_C A}(K, A) \cong \Delta \oplus \text{Hom}_{A \otimes_C A}(K, A). \]

By tracing these isomorphisms through, one checks that the composite is the identity map on \( \Delta \). Hence

\[ \text{Hom}_{A \otimes_C A}(K, A) = \text{Hom}_{A,A}(K, A) = (0). \]

((3) \( \Rightarrow \) (2)) Writing \( A \otimes_R A = K \oplus M, M \oplus B \cong A^n \), we get an \( A \), \( A \)-map...
from $A \otimes_R A$ into $A^n$ by projecting $A \otimes_R A$ onto $M$ and injecting $M$ into $A^n$. Let $1 \otimes 1 \mapsto u \in M$, $u \mapsto (d_i) \in A^n$. Note that $1 \otimes r = r \otimes 1$, all $r \in R$, implies $d_i \in \Delta$, each $i$. Let $e_i \in A^n$ be the element whose $i$th coordinate is $1 \in A$ and whose other coordinates are zero. Let $m_i + b_i \mapsto e_i$ under the isomorphism $M \oplus B \to A^n$. Then $\sum d_i m_i + d_i b_i \mapsto (d_i)$. Thus $u - \sum d_i m_i - \sum d_i b_i = 0$ in $A^n$. It follows that $u - \sum d_i m_i = 0$ in $M$, and $\sum d_i b_i = 0$ in $B$.

Under the projection $A^n \to M$, $e_i \mapsto m_i$; so $a e_i = e_i a = am_i = m_i a$, all $a \in A$. Thus $m_i \in (A \otimes_R A)^A$, all $i$. Write $m_i = \sum a_{ij} \otimes b_{ij} \in A \otimes_R A$. Then $1 \otimes 1 - u \in K$, and $1 \otimes 1 - u = 1 \otimes 1 - \sum d_i m_i = 1 \otimes 1 - \sum d_i a_{ij} \otimes b_{ij}$.

Now $K \to A \otimes_R A \to \text{Hom}_C(\Delta, A) \to A^n$, and the first and third maps are injective. Since $\text{Hom}_A(K, A) = (0)$, we must have $K \subseteq \ker(\phi)$. Therefore $0 = \phi(1 \otimes 1 - u) = \phi(1 \otimes 1) - \phi(\sum d_i a_{ij} \otimes b_{ij})$. This says $d = \sum d_i a_{ij} \otimes b_{ij}$, all $d \in \Delta$.

((2) $\Rightarrow$ (1)) Note that $\sum a_{ij} \otimes b_{ij} \in (A \otimes_R A)^A$ implies $\sum a_{ij} \otimes b_{ij} \in C$, for all $i$ and all $d \in \Delta$. Let $f_i \in \text{Hom}_C(\Delta, C)$ be defined by $f_i(d) = \sum a_{ij} \otimes b_{ij}$, $d \in \Delta$, for each $i$. Then $d = \sum f_i(d_i)$, all $d \in \Delta$. It is well-known that this implies $\Delta_C$ is finitely generated and projective.

Define $\phi : \text{Hom}_C(\Delta, A) \to A \otimes_R A$ by $f \mapsto \sum f(d_i) a_{ij} \otimes b_{ij}$. For all $a \in A$,

$\phi(af) = \sum_{i,j} a f(d_i) a_{ij} \otimes b_{ij} = a \phi(f)$, and

$\phi(fa) = \sum_{i,j} f(a) d_i a_{ij} \otimes b_{ij} = \sum_{i,j} f(d_i) a a_{ij} \otimes b_{ij}

= \sum_{i,j} f(d_i) a_{ij} \otimes b_{ij} a = \phi(f) a$.

Hence $\phi$ is an $A$, $A$-map. Furthermore,

$\phi \phi(f)(d) = \sum_{i,j} f(d_i) a_{ij} \otimes b_{ij} = f(\sum_{i,j} d_i a_{ij} \otimes b_{ij}) = f(d)$,

since $\sum a_{ij} \otimes b_{ij} \in C$, all $i$. Hence $\phi \circ \phi$ is the identity map on $\text{Hom}_C(\Delta, A)$; i.e. $\phi$ splits the exact sequence $A \otimes_R A \xrightarrow{\phi} \text{Hom}_C(\Delta, A) \to (0)$. It follows that $A$ is strongly separable over $R$, and the Theorem is proved.

We are indebted to K. Sugano for an example of a separable ring extension that is not strongly separable. The following example is a variant of the one which he provided.

**Example 3.6.** Let $G = \{e, \beta, \beta^2\}$, a three element group, and let $K$ be the Galois field with three elements. Let $S = KG$, the group algebra of $G$ over $K$. For $s = ae + b \beta + c \beta^2 \in S$, define $\bar{s} = ae + b \beta^2 + c \beta$. The map $s \mapsto \bar{s}$ is an automorphism of $S$. 

Let $A$ be the $2 \times 2$ matrix ring over $S$, $R=\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} | s \in S \subseteq A$, and $T=\{a(e+\beta+\beta^2)|a \in K\} \subseteq S$. Since $S$ is commutative, the center $C$ of $A$ is $C=\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} | s \in S \}$. It is straightforward to verify that the centralizer $\Delta$ of $R$ in $A$ is $\Delta=\begin{bmatrix} s_1 & t_1 \\ t_2 & s_2 \end{bmatrix} | s_1, s_2 \in S, t_1, t_2 \in T \}$. Let $\sigma=e+\beta+\beta^2 \subseteq S$. As $C$-module, $C\begin{bmatrix} 0 & \sigma \\ 0 & 0 \end{bmatrix}$ is a direct summand of $\Delta$. Thus if $\Delta_C$ were projective, $C\begin{bmatrix} 0 & \sigma \\ 0 & 0 \end{bmatrix}$ would be also, and $S\sigma$ would be projective as $S$-module. That this is not the case is seen as follows.

Let $\varepsilon: S \rightarrow S\sigma$, $s \mapsto \sigma s$, a surjective map of $S$-modules. If $S\sigma$ is projective, there is a splitting map $\tau$. If $\tau(\sigma)=u, S=\ker(\varepsilon) \oplus Su$. Then $\varepsilon(\sigma u)=\sigma \varepsilon(u)=\sigma \tau(\sigma)=\sigma^2$. But $\sigma^2=0$. So $\sigma u \in \ker(\varepsilon) \cap Su=(0)$; $\sigma u=0$. Then $u \in \ker(\varepsilon) \cap Su$; i.e. $u=0$, a contradiction.

Since $\Delta_C$ is not projective, $A$ is not strongly separable over $R$. However $A$ is separable over $R$. The element $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is in the $A$-center of $A \otimes_R A$ and is mapped to the unity element of $A$ by $\mu: A \otimes_R A \rightarrow A$.

We now return to our general setting where $R$ is a subring of $A$, $\Delta$ is the centeralizer of $R$ in $A$ and $\phi: A \otimes_R A \rightarrow \text{Hom}_C(\Delta, A)$. The proof of the following Lemma is straightforward and is omitted.

**Lemma 3.7.** Let $K=\ker(\phi)$ and $S=\{s \in A|s \otimes 1-1 \otimes s \in K\}$. Then $S$ and $\Delta$ are centralizers of each other in $A$.

With $S$ as defined in the Lemma we have

**Proposition 3.8.** If $A$ is strongly separable over $R$ then $A$ is strongly separable over $S$. If $S$ is separable over $R$ then $S$ is strongly separable over $R$.

**Proof.** Since $A^S=\Delta$, and $\Delta_C$ is finitely generated and projective by hypothesis, to prove the first statement we need only show that the map $\phi: A \otimes S A \rightarrow \text{Hom}_C(\Delta, A)$ splits. Let $\phi'$ be the splitting map of $\phi: A \otimes_R A \rightarrow \text{Hom}_C(\Delta, A)$, and let $f: A \otimes_R A \rightarrow A \otimes S A$ be the natural map defined because $R \subseteq S$. Then $f \circ \phi'$ is a splitting map for $\phi$.

Assume $S$ is separable over $R$ and let $C'$ denote the center of $S$. Then $C'=\Delta \cap S$, since $A^S=\Delta$. Furthermore, $\Delta'=S'^S=C'$. So $\Delta_{C'}$ is trivially finitely generated and projective. Also $\text{Hom}_{C'}(\Delta', S) \cong S$; so the splitting of $S \otimes_R S \rightarrow \text{Hom}_{C'}(\Delta', S)$ is equivalent to the splitting of $S \otimes_R S \rightarrow S \rightarrow (0)$. 

The following proposition is the analogue for strong separability of 1.5 of \[12\].

**Proposition 3.9.** If \( A \) is strongly separable over \( R \), then each of the following maps is a split monomorphism:

(i) \( \Delta \otimes_{C} A \to \text{Hom}(RA, RA), \ d \otimes a \mapsto [x \mapsto dx a] \),

(ii) \( A \otimes_{C} \Delta \to \text{Hom}(AR, AR), \ a \otimes d \mapsto [x \mapsto ax d] \),

(iii) \( \Delta \otimes_{C} \Delta \to \text{Hom}_{R, R}(A, A), \ d_{1} \otimes d_{2} \mapsto [x \mapsto d_{1} xd_{2}] \).

**Proof.** (i) Using Lemma 3.4 we obtain \( \Delta \otimes_{C} A \cong \text{Hom}(\Delta, A) \). Since \( A \) is strongly separable, \( A \otimes_{R} A \cong K \oplus \text{Hom}(\Delta, A) \). Applying these isomorphisms and the Adjoint Functor Theorem, we have

\[
\text{Hom}(RA, RA) \cong \text{Hom}(\Delta, A) \cong \text{Hom}(\Delta, A) \otimes_{C} A \cong \Delta \otimes_{C} A.
\]

where the last map \( \pi \) is the projection map arising from the direct sum decomposition. Tracing through these maps one checks that the composite is a splitting map for the map in (i).

(ii) The proof is similar to the proof of (i).

(iii) In the proof of part (i) above the isomorphism of \( \text{Hom}(RA, RA) \) onto \( \text{Hom}(\Delta, A) \) maps \( \text{Hom}(RA, RA) \) onto \( \text{Hom}(\Delta, A) \). So we have

\[
\text{Hom}(RA, RA) \cong \Delta \otimes_{C} \Delta \cong \text{Hom}(\Delta, A) \otimes_{C} \Delta \cong \text{Hom}(\Delta, A).
\]

This proves (iii).

**Proposition 3.10.** Assume \( A \) is strongly separable over \( R \), \( A \otimes_{R} A \cong \text{Hom}(\Delta, A) \). Then for every \( A \)-bimodule \( M \), \( M^{R} \cong A \otimes_{C} M^{A} \oplus \text{Hom}_{A, A}(K, M) \). In particular, \( (A \otimes_{R} A)^{R} \cong A \otimes_{C} (A \otimes_{R} A)^{A} \oplus \text{Hom}_{A, A}(K, M) \).

**Proof.** From Lemma 3.4 we have

\[
A \otimes_{C} M^{A} \cong A \otimes_{C} \text{Hom}_{A, A}(A, M) \cong \text{Hom}_{A, A}(\text{Hom}(\Delta, A), M).
\]

Then

\[
M^{R} \cong \text{Hom}_{A, A}(A \otimes_{R} A, M) \cong \text{Hom}_{A, A}(\text{Hom}(\Delta, A), M) \oplus \text{Hom}_{A, A}(K, M)
\cong A \otimes_{C} M^{A} \oplus \text{Hom}_{A, A}(K, M).
\]
4. Automorphisms. If $\sigma$ is an automorphism of a ring $A$ we let $A_\sigma$ denote the $A$, $A$-bimodule such that as left $A$-module $A_\sigma$ is just $A$, but where the right module structure is “twisted” by $\sigma$, $x \cdot a = x \sigma(a)$ for $x \in A_\sigma$, $a \in A$.

If $R$ is a subring of $A$ then $A$ is a Galois extension of $R$ if there is a finite group $G$ of automorphisms of $A$ such that $R = A^G$, and such that there exists $x_i$, $y_i$, $1 \leq i \leq n$, for which

$$\sum_i x_i \sigma(y_i) = \begin{cases} 0 & \text{if } \sigma \neq 1 \\ 1 & \text{if } \sigma = 1. \end{cases}$$

If $G$ is a finite group of $R$-automorphisms of $A$ there is an $A$, $A$-bimodule map $h : A \otimes_R A \rightarrow AG$, defined by $a \otimes b \mapsto \sigma(a) \sigma(b).$ Here, $AG$ is the twisted group algebra of $G$ over $A$. It can be shown that if $R = A^G$ then $A$ is a Galois extension of $R$ if and only if $h$ is an isomorphism.

The twisted group algebra $AG$ is a direct sum $\bigoplus_{\sigma \in G} A \sigma$, and, for each $\sigma$, $A \sigma$ is $A$, $A$-bimodule isomorphic to $A_\sigma$. Thus when $A$ is a Galois extension of $R$ with Galois group $G$, $A \otimes_R A \cong \bigoplus_{\sigma \in G} A \sigma$. This motivates the following definition.

**Definition 4.1.** $A$ is a pseudo-Galois extension of $R$ if there is a finite set $S$ of $R$-automorphisms of $A$ and a positive integer $n$ such that $A \otimes_R A$ is isomorphic to a direct summand of $\bigoplus_{\sigma \in S} A_\sigma^n$.

If in [Definition 4.1] $S = \{1\}$, then the condition is that $A \otimes_R A$ is isomorphic to a direct summand of $A^n$, for some positive integer $n$. This is just the condition that $A$ be $H$-separable over $R$. Thus $H$-separable extensions and Galois extensions are pseudo-Galois.

In Definition 4.1 we will assume that $A_\sigma \neq A$, if $\sigma \neq \tau$, $\sigma$, $\tau \in S$.

Assume that $\sigma$ and $\tau$ are automorphisms of a ring $A$ and let $\mu : A_\sigma \rightarrow A_\tau$ be an $A$, $A$-bimodule map. Let $\mu(1) = x$. Then, for each $a \in A$,

$$\sigma(a) x = \sigma(a) \mu(1) = \mu \sigma(a) = \mu(1 \cdot a) = \mu(1) \cdot a = x \tau(a).$$

Conversely, if $x \in A$ such that $\sigma(a) x = x \tau(a)$ for all $a \in A$, there is a unique bimodule map $\mu : A_\sigma \rightarrow A_\tau$ such that $\mu(1) = x$. The map $\mu$ is an isomorphism if and only if $x$ is a unit in $A$, and in this case $\tau \sigma^{-1}$ is an inner automorphism of $A$. Conversely, if $\tau \sigma^{-1}(a) = a^{-1} x a$, for some unit $x$ in $A$ then $\sigma(a) x = x \tau(a)$ and there is a unique isomorphism $\mu : A_\sigma \rightarrow A_\tau$, such that $\mu(1) = x$. Let

$$J_{\sigma, \tau} = \{ x \in A | \sigma(a) x = x \tau(a), \text{ all } a \in A \} .$$

Then $J_{\sigma, \tau}$ is a $C$-module and $J_{\sigma, \tau} \cong \text{Hom}_{A, A}(A_\sigma, A_\tau)$; $A_\sigma \cong A$, if and only if
\sigma^{-1} is an inner automorphism of \( A \). We will denote \( J_{1,\sigma} \) by \( J_{\sigma} \). Then \( J_{\sigma,\tau} = J_{\sigma^{-1}} \).

In part of what follows we will assume that any nonzero \( A, A \)-bimodule map from \( A_{\sigma} \) to \( A_{\tau} \) is an automorphism. This condition holds, for example, if \( A \) is a simple ring.

**Proposition 4.2.** Let \( A \) be a pseudo-Galois extension of \( R \), and assume that if \( \sigma, \tau \in \text{Aut}_R(A) \) that any nonzero bimodule map from \( A_{\sigma} \) to \( A_{\tau} \) is an isomorphism. Then

(i) \( \text{Hom}_C(\Delta, A_{\tau}) \) is isomorphic to a direct summand of \( A_{\tau}^{\sigma} \), each \( \sigma \in S \).

(ii) If \( I \) is the group of inner \( R \)-automorphisms of \( A \) then \( S \) contains exactly one element from each coset of \( I \) in \( \text{Aut}_R(A) \).

(iii) \( A \otimes_R A \cong \sum_{\sigma \in S} \oplus \text{Hom}_C(\Delta, A_{\sigma}) \).

**Proof.** Let \( \sum_{\sigma \in S} \oplus A_{\sigma}^{\sigma} \cong A \otimes_R A \oplus B \). Then

\[
\text{Hom}_{A,A}(A \otimes_R A, A) \oplus \text{Hom}_{A,A}(B, A) \cong \text{Hom}_{A,A}(\sum_{\sigma \in S} \oplus A_{\sigma}^{\sigma}, A) ;
\]

i.e. \( \Delta \oplus B \cong \sum_{\sigma \in S} \oplus \text{Hom}_{A,A}(A_{\sigma}^{\sigma}, A) \). There must exist \( \sigma_0 \in S \) such that \( \text{Hom}_{A,A}(A_{\sigma_0}^{\sigma}, A) \neq (0) \). Hence \( A_{\sigma_0}^{\sigma} \cong A_{\tau}^{\sigma} \), and \( \text{Hom}_{A,A}(A_{\sigma_0}^{\sigma}, A) \cong C^{\sigma} \). For \( \sigma \neq \sigma_0 \), \( \text{Hom}_{A,A}(A_{\sigma_0}^{\sigma}, A) = (0) \); hence \( \Delta \oplus B \cong C^{\sigma} \).

Let \( \tau \in \text{Aut}_R(A) \). Then \( A_{\sigma}^{\sigma} \cong \text{Hom}_C(C^{\sigma}, A_{\tau}) \cong \text{Hom}_C(\Delta, A_{\tau}) \oplus \text{Hom}_C(B', A_{\tau}) \).

Define \( \phi : A \otimes_R A \to \text{Hom}_C(\Delta, A_{\tau}) \) by \( \phi(a \otimes b) = [d \mapsto ad \tau(b)] \). Then \( \phi \) is an \( A, A \)-map, where \( \text{Hom}_C(\Delta, A_{\tau}) \) is an \( A, A \)-bimodule via the action on \( A_{\tau} \). We then have a sequence of bimodule maps

\[
\sum_{\sigma \in S} \oplus A_{\sigma}^{\sigma} \longrightarrow A \otimes_R A \xrightarrow{\phi_{\tau}} \text{Hom}_C(\Delta, A_{\sigma}) \longrightarrow A_{\sigma}^{\sigma} ,
\]

whose composition is nonzero. Hence there exists \( \sigma' \in S \) such that \( A_{\sigma'} \cong A_{\tau} \). Then \( S \) contains exactly one element from each coset of \( I \) in \( \text{Aut}_R(A) \).

Let \( \beta' \) denote the split injective mapping of \( A \otimes_R A \) to \( \sum_{\sigma \in S} \oplus A_{\sigma}^{\sigma} \) assumed to exist because \( A \) is a pseudogaits extension of \( R \), and let \( \beta^{\prime} \) denote the splitting map. For each \( \tau \in S \) let \( u_\tau, v_\tau \in A \otimes_R A \) be chosen so that \( \beta(1 \otimes 1) = u_\tau' + v_\tau' \), where \( u_\tau' \in A_{\tau}^{\sigma} \) and \( v_\tau' \in \sum_{\sigma \neq \tau} \oplus A_{\sigma}^{\sigma} \) and such that \( u_\tau = \beta'(u_\tau') \) and \( v_\tau = \beta'(v_\tau') \). Since \( \text{Hom}_C(\Delta, A_{\tau}) \) is isomorphic to a sub-bimodule of \( A_{\tau}^{\sigma} \) we must have \( \phi_{\tau}(v_\tau) = 0 \). Thus \( \phi_{\tau}(1 \otimes 1) = \phi_{\tau}(u_\tau) \).

For \( 1 \leq i \leq n \) let \( e_i \) denote the element of \( A_{\tau}^{\sigma} \) whose \( i^{th} \) coordinate is 1 and whose \( j^{th} \) coordinate is 0, \( j \neq i \), \( 1 \leq j \leq n \). Then \( u_\tau' = \sum_{i=1}^{n} d_i e_i, d_i \in A \). For each \( r \in R, r \otimes 1 = 1 \otimes r \); so \( \sum_{i=1}^{n} rd_i e_i = \sum_{i=1}^{n} d_i r e_i \). Thus \( rd_i = d_i r, 1 \leq i \leq n \). It
follows that \( d_i \in \Delta \), each \( i \).

Let \( h'(e_i) = \sum_j a_{ij} \otimes b_{ij} \in A \otimes_R A \), \( 1 \leq i \leq n \). Note that \( \tau(a) e_i = e_i \ast a \) implies that \( \tau(a) \sum_j a_{ij} \otimes b_{ij} = \sum_j a_{ij} \otimes b_{ij} a \), all \( a \in A \). Mapping with \( \phi_i \), we obtain

\[
\tau(a) \sum_j a_{ij} d \tau(b_{ij}) = \sum_j a_{ij} d \tau(b_{ij}) \tau(a), \text{ all } a \in A, \ d \in \Delta.
\]

Thus \( \sum_j a_{ij} d \tau(b_{ij}) \in C \), all \( d \in \Delta \). Now \( \phi_i(1 \otimes 1) = \phi_i(u) = \phi_i(\sum_i d_i a_{ij} \otimes b_{ij}). \)

Hence \( d = \sum_i d_i a_{ij} \otimes b_{ij} \), all \( d \in \Delta \).

Let us now define \( \phi_i : \text{Hom}_C(\Delta, A) \to A \otimes_R A \) by \( \psi_i(f) = \sum_{i,j} f(d_{ij}) a_{ij} \otimes b_{ij} \).

\( \phi_i \) is clearly a left \( A \)-module map. If \( a \in A \), then

\[
\psi_i(\tau(a)f) = \sum_{i,j} (\tau(a)f)(d_{ij}) a_{ij} \otimes b_{ij} = \sum_{i,j} f(d_{ij}) \tau(a)a_{ij} \otimes b_{ij} = \psi_i(f) a.
\]

Hence \( \phi_i \) is a bimodule map.

We now show that \( \psi_i \) splits \( \phi_i \). Since \( \sum_j a_{ij} d \tau(b_{ij}) \in C \), \( 1 \leq i \leq n \); for each \( f \in \text{Hom}_C(\Delta, A_i) \) we have

\[
\sum_{i,j} f(d_{ij}) a_{ij} d \tau(b_{ij}) = f(\sum_{i,j} d_i a_{ij} \otimes b_{ij}) = f(d).
\]

Then \( \phi_i \circ \psi_i \) is the identity map on \( \text{Hom}_C(\Delta, A_i) \). Also, it is straightforward that \( \phi_i \circ \phi_i(\tau(a)) = \tau(a) \). Since \( k(1 \otimes 1) = \sum_{\tau \in \Sigma} u' \), we have \( 1 \otimes 1 = \sum \tau \ast u = \sum \phi_i \circ \phi_i (u) = \sum \phi_i \circ \phi_i (1 \otimes 1) \); and thus \( \phi_i \circ \phi_i \) is the identity map on \( A \otimes_R A \).

Therefore \( A \otimes_R A \cong \sum_{\sigma \in \Sigma} \oplus \text{Hom}_C(\Delta, A_i) \).

**Corollary 4.3.** Let \( A \) be a pseudo-Galois extension of \( R \), and assume for each \( \sigma, \tau \in \text{Aut}_R(A) \) that any nonzero bimodule map from \( A_\sigma \) to \( A_\tau \) is an isomorphism. Then \( A \) is strongly separable over \( R \).

**Proof.** Assume \( A \otimes_R A \cong \sum_{\sigma \in \Sigma} \oplus \text{Hom}_C(\Delta, A_i) \). Let \( K = \sum_{\sigma \in \Sigma} \oplus \text{Hom}_C(\Delta, A_i) \), and apply **Theorem 3.5**.

We have seen that \( H \)-separable extensions are pseudo-Galois. There appears to be no general relationship between strongly separable extensions and pseudo-Galois extensions. The following proposition for strongly separable extensions has a conclusion similar to, but weaker than, that of Proposition 4.2.

Recall that for each \( R \)-automorphism \( \sigma \) of \( A \), \( \phi_\sigma : A \otimes_R A \to \text{Hom}_C(\Delta, A_i) \) is defined by \( \phi_\sigma(a \otimes b) = [d \mapsto a \sigma(d) b] \). Also, let \( I \) denote the subgroup of inner automorphisms in \( \text{Aut}_R(A) \).
Proposition 4.4. Let $A$ be a strongly separable extension of $R$. Then for each $R$-automorphism $\sigma$ of $A$ the map $\check{\phi}$ is a split epimorphism. Assume further that $I$ is of finite index in $\text{Aut}_R(A)$ and that if $\sigma$ and $\tau$ are $R$-automorphisms of $A$ then any nonzero bimodule map from $A_\sigma$ to $A_\tau$ is an isomorphism. Then there exists a set $S$ of $R$-automorphisms of $A$ containing exactly one element from each coset of $I$ in $\text{Aut}_R(A)$ such that $\sum_{\sigma \in S} \text{Hom}_C(\Delta, A_\sigma)$ is isomorphic to a direct summand of $A \otimes_R A$.

Proof. As in the proof of Theorem 3.5, we can find elements $d_i \in \Delta$, $a_{ij}, b_{ij} \in A$ such that for each $d \in \Delta$, $d = \sum d_i a_{ij} b_{ij}$, and such that $\sum a_{ij} \otimes b_{ij} \in (A \otimes_R A)^A$ and $\sum a_{ij} b_{ij} \in C$, each $i$. We define $\phi_\sigma : \text{Hom}_C(\Delta, A_\sigma) \to A \otimes_R A$ by $\phi_\sigma(f) = \sum_{i,j} f(d_i) a_{ij} \otimes \sigma^{-1}(b_{ij})$. $\phi_\sigma$ is clearly a map of left $A$-modules.

We note that $1 \otimes \sigma^{-1}$ is a well-defined map of abelian groups from $A \otimes_R A$ to $A \otimes_R A$. Since, for each $a \in A$,

$$\sum_j \sigma(a) a_{ij} \otimes b_{ij} = \sum_j a_{ij} \otimes b_{ij} \sigma(a),$$

we can apply $1 \otimes \sigma^{-1}$ to obtain

$$\sum_j \sigma(a) a_{ij} \otimes \sigma^{-1}(b_{ij}) = \sum_j a_{ij} \otimes \sigma^{-1}(b_{ij}) a, \text{ each } i.$$ We now show that $\phi_\sigma$ is a right $A$-module map. For each $a \in A$, $f \in \text{Hom}_C(\Delta, A_\sigma)$,

$$\phi_\sigma(fa) = \sum_{i,j} f(d_i) a_{ij} \otimes \sigma^{-1}(b_{ij}) = \sum_{i,j} f(d_i) \sigma(a) a_{ij} \otimes \sigma^{-1}(b_{ij})$$

$$= \sum_{i,j} f(d_i) a_{ij} \otimes \sigma^{-1}(b_{ij}) a = \phi_\sigma(f) a .$$

Next we show that $\phi_\sigma$ splits $\phi_*$. For each $f \in \text{Hom}_C(\Delta, A_\sigma)$,

$$\phi_* \circ \phi_\sigma(f)(d) = \sum_{i,j} f(d_i) a_{ij} d \sigma(\sigma^{-1}(b_{ij})) = \sum_{i,j} f(d_i) a_{ij} b_{ij}$$

$$= f(\sum_{i,j} d_i a_{ij} b_{ij}) = f(d).$$

Now, assume that if $\sigma$ and $\tau$ are $R$-automorphisms of $A$ then any nonzero bimodule map from $A_{\sigma}$ to $A_{\tau}$ is an isomorphism.

For each $\sigma \in \text{Aut}_R(A)$ we write $A \otimes_R A = K_\sigma \oplus L_\sigma$ where $L_\sigma \cong \text{Hom}_C(\Delta, A_\sigma)$. Since $\Delta_C$ is finitely generated and projective, $L_\sigma$ is isomorphic to a direct summand of $A^{n_\sigma}$, some positive integer $n$. If $\sigma, \tau \in \text{Aut}_R(A)$ and $A_\sigma \not\cong A_\tau$, then there is no nonzero bimodule map from $A_\sigma$ to $A_\tau$; hence $L_\sigma \subseteq K_\sigma$.

Let $S$ be a subset of $\text{Aut}_R(A)$ containing exactly one element from each coset of $I$, and let $K' = \bigcap_{\sigma \in S} K_\sigma$. Then $A \otimes_R A = \sum_{\sigma \in S} L_\sigma \oplus K'$. This completes the proof of the Proposition.
We now drop the hypothesis that for \(\sigma, \tau \in \text{Aut}_R(A)\), any nonzero bimodule map from \(A_e\) to \(A_e\) is an isomorphism. Recall that for \(\sigma \in \text{Aut}_R(A)\), 
\[J_e=\{x \in A \mid ax = x\sigma(a), \text{ for all } a \in A\}\]. 
\(J_e\) is a \(C\)-module and \(\text{Hom}_{A,A}(A, A_e) \cong J_e\) under the map \(f \mapsto f(1)\).

Hirata [3] has shown that if \(A\) is an \(H\)-separable extension of \(R\) then \(J_e\) is finitely generated and projective of rank 1, and is free if and only if \(\sigma\) is an inner automorphism. In the following we generalize to strongly separable extensions.

We assume throughout the rest of this section that \(A\) has no nontrivial central idempotents.

**Proposition 4.5.** Assume \(A\) is strongly separable over \(R\); i.e. \((0) \rightarrow K \rightarrow A \otimes_R A \rightarrow \text{Hom}_C(\Delta, A) \rightarrow (0)\) is a split exact sequence. Then \(J_e \neq (0)\) if and only if \(\text{Hom}_{A,A}(K, A_e) = (0)\). In this case \(J_e\) is finitely generated and projective of rank 1.

**Proof.** From Lemma 3.4,
\[\text{Hom}_{A,A}(\Delta, A, A_e) \cong \Delta \otimes_C \text{Hom}_{A,A}(A, A_e) \cong \Delta \otimes_C J_e.\]
Hence
\[J_e \cong \text{Hom}_{A,A}(A \otimes_R A, A_e) \cong \text{Hom}_{A,A}(\text{Hom}_C(\Delta, A, A_e) \oplus \text{Hom}_{A,A}(K, A_e))\]
\[\cong \Delta \otimes_C J_e \oplus \text{Hom}_{A,A}(K, A_e).\]
From this we see first that \(J_e \neq (0)\) implies \(\text{Hom}_{A,A}(K, A_e) \neq (0)\). Next since \(C\) is a direct summand of \(\Delta\), we conclude from the above that \(J_e\) is a direct summand of \(\Delta\). So \(J_e\) is finitely generated and projective. Finally let \(t = \text{rank}(J_e)\), \(n = \text{rank}(\Delta)\). Then, from the above, we have \(n = n \cdot t + \text{rank}(\text{Hom}_{A,A}(K, A_e))\). If \(J_e \neq (0)\), we must have \(t = 1\) and \(\text{Hom}_{A,A}(K, A_e) = (0)\). This completes the proof.

Let \(\sigma \in \text{Aut}_R(A)\). Then \(\sigma = 1 \otimes \sigma: A \otimes_R A \rightarrow A \otimes_R A\) is an automorphism of \(A\) as left \(A\)-module, but is not a bimodule map in general. However, if \(M\) is a sub-bimodule of \(A \otimes_R A\) then \(\sigma(M)\) is again a sub-bimodule.

Now assume \(A\) is a strongly separable extension of \(R\). For each \(\sigma \in \text{Aut}_R(A)\), \(A \otimes_R A = K_\sigma \oplus L_\sigma\), where \(L_\sigma \cong \text{Hom}_C(\Delta, A_e)\) and \(K_\sigma = \ker(\phi_\sigma)\). Let \(L = L_1\).

**Lemma 4.6.** Assume \(A\) is a strongly separable extension of \(R\), and let \(\sigma \in \text{Aut}_R(A)\). If \(\text{Hom}_{A,A}(A, A_e) \neq (0)\), then \(\text{Hom}_{A,A}(A_e, A) \neq (0)\); so \(J_{e-1} \neq (0)\). Further, \(\text{Hom}_{A,A}(A_e, A) \neq (0)\) implies \(K = K_\sigma, L \cong L_\sigma\).

**Proof.** If \(\text{Hom}_{A,A}(A, A_e) \neq (0)\) then \(\text{Hom}_{A,A}(K, A_e) = (0)\), by Proposition 4.5. Hence the projection of \(K\) to \(L\) arising from the direct sum decomposition \(A \otimes_R A = K_\sigma \oplus L_\sigma\) must be the zero map, since \(L_\sigma\) is isomorphic to a
direct summand of $A_\sigma^n$, for some positive integer $n$. Thus $K \subseteq K_\sigma$. The
projection of $L_\sigma$ to $L$ arising from the direct sum decomposition $A \otimes_R A = K \oplus L$ must be nonzero. Since $L$ is isomorphic to a direct summand of $A^n$, for some positive integer $m$, this gives rise to a nonzero bimodule map from $A_\sigma$ to $A$. Thus $J_\sigma-1 \cong \text{Hom}_{A_\sigma}(A_\sigma, A) \neq (0)$. The argument showing $K \subseteq K_\sigma$ can now be used to show $K_\sigma \subseteq K$, giving $K = K_\sigma$. Thus $A \otimes_R A \cong K \oplus L \cong K_\sigma \oplus L_\sigma = K \oplus L_\sigma$, from which it follows that $L \leq L_\sigma$.

**Proposition 4.7.** Let $A$ be a strongly separable extension of $R$ and let $\sigma \in \text{Aut}_R(A)$ such that $J_\sigma \neq (0)$. Then $J_\sigma^{-1} \neq (0)$ and $J_\sigma^{-1} \cong J_\sigma^{-1} = \text{Hom}_C(J_\sigma, C)$.

**Proof.** Let $\beta : L \rightarrow L_\sigma$ be an isomorphism, guaranteed by the Lemma, and let $\alpha = \beta^{-1}$. Since $C$ is isomorphic to a direct summand of $\underline{\delta}$, $A$ is isomorphic to a direct summand of $\text{Hom}_C(A, A) \cong L$. Hence there exist maps $\gamma : A \rightarrow L$ and $\delta : L \rightarrow A$ such that $\delta \circ \gamma = 1_A$. $L_\sigma$ is isomorphic to a direct summand of $A_\sigma^n$, some positive integer $n$; so there exist maps $f_i : L_\sigma \rightarrow A_\sigma$, $g_i : A_\sigma \rightarrow L_\sigma$, $1 \leq i \leq n$, such that $\sum g_i \circ f_i = 1_{L_\sigma}$.

Let $\bar{f}_i = f_i \beta \gamma : A \rightarrow A_\sigma$, $\bar{g}_i = \delta \sigma g_i : A_\sigma \rightarrow A$. Then $\sum \bar{g}_i \circ \bar{f}_i = 1_A$. Thus the map

$J_\sigma^{-1} \otimes_C J_\sigma \cong \text{Hom}_{A_\sigma}(A_\sigma, A) \otimes_C \text{Hom}_{A_\sigma}(A, A) \rightarrow \text{Hom}_{A_\sigma}(A, A) \cong C$,

defined by $g \otimes f \mapsto g \circ f$, for $g \in \text{Hom}_{A_\sigma}(A_\sigma, A)$ and $f \in \text{Hom}_{A_\sigma}(A, A)$, is surjective. It follows that $J_\sigma^{-1} \cong J_\sigma^*$. This completes the proof.

We summarize the above as follows. Let

$G = \left\{ \sigma \in \text{Aut}_R(A) \bigg| J_\sigma \neq (0) \right\} = \left\{ \sigma \in \text{Aut}_R(A) \right\}$.

Then $G$ is a subgroup of $\text{Aut}_R(A)$. If $\sigma, \tau \in \text{Aut}_R(A)$ then $\text{Hom}_{A_\sigma}(A_\sigma, A_\tau) \cong \text{Hom}_{A_\sigma}(A, A_\sigma) \neq (0)$ if and only if $\sigma$ and $\tau$ are in the same right coset modulo $G$.

**References**


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