Additional Structure on the Category of Mackey Functors

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Goal (Broadest Sense)

Understand the underlying algebra of Mackey functors and Tambara functors.

Develop a concrete structure on the category of $G$-Mackey functors that provides a nice characterization of $G$-Tambara functors.

$G = \text{cyclic group of order } p^n (p \text{ prime})$
Overview

Goal (Brodest Sense)

- Understand the underlying algebra of Mackey functors and Tambara functors.

\[ G = \text{cyclic group of order } p^n, \quad p \text{ prime} \]
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- Understand the underlying algebra of Mackey functors and Tambara functors.
- Develop a concrete structure on the category of $G$-Mackey functors that provides a nice characterization of $G$-Tambara functors.

$G =$ cyclic group of order $p^n$ ($p$ prime)
Let $G$ be a finite abelian group.
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$k^{th}$ Stable Homotopy Group of $X$, $\pi_k(X)$, $k \in \mathbb{Z}$
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**$k^{th}$ Stable Homotopy Group of $X$, $\pi_k(X)$, $k \in \mathbb{Z}$**

- For all $H \leq G$ define

\[
\pi_k^H(X) = [S^k, X]^H
\]
Let $G$ be a finite abelian group. Let $X$ be a $G$-spectrum.

$k^{th}$ Stable Homotopy Group of $X$, $\pi_k(X)$, $k \in \mathbb{Z}$

- For all $H \leq G$ define

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- The collection of all $\pi^H_k(x)$ is a Mackey functor
Let $G$ be a finite abelian group. Let $X$ be a $G$-spectrum.

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- For all $H \leq G$ define
  \[ \pi^H_k(X) = [S^k, X]^H \]
- The collection of all $\pi^H_k(X)$ is a Mackey functor
- Define $\pi_k(X)$ to be this Mackey functor.
A $G$-Mackey functor $M$ consists of a pair of functors

$$(M_*, M^*) : \mathcal{S}et^\text{Fin}_G \to \mathcal{A}b$$

such that

- $M^*(X) = M_*(X) := M(X)$ for all $X$ in $\mathcal{S}et^\text{Fin}_G$
- $M(X \amalg Y) = M(X) \oplus M(Y)$
- If the diagram below is a pullback diagram in $\mathcal{S}et^\text{Fin}_G$ then $M_*(h) \circ M^*(f) = M^*(g) \circ M_*(k)$ in $\mathcal{A}b$.

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow{f} & & \downarrow{g} \\
Z & \xrightarrow{k} & W
\end{array}
\]
A $C_p$-Mackey Functor $M$

$C_p$ - Cyclic group of prime order
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$C_p$ - Cyclic group of prime order

$M(C_p/C_p)$

$M(C_p/e)$
A $C_p$-Mackey Functor $\underline{M}$

$C_p$ - Cyclic group of prime order

$\underline{M}(C_p/C_p) \xrightarrow{\text{transfer} = tr} \underline{M}(C_p/e)$
A $C_p$-Mackey Functor $\underline{M}$

$C_p$ - Cyclic group of prime order

$\underline{M}(C_p/C_p)$

restriction = $res$

$\underline{M}(C_p/e)$

transfer = $tr$
$C_p$ - Cyclic group of prime order

$\gamma$ generates the Weyl group, $W_{C_p}(e)$
$A$ $C_p$-Mackey Functor $\underline{M}$

$C_p$ - Cyclic group of prime order

\[ \underline{M}(C_p/C_p) \]

\[ \underline{M}(C_p/e) \]

\[ \text{restriction} = res \]

\[ \text{transfer} = tr \]

- $\gamma$ generates the Weyl group, $W_{C_p}(e)$
- $\text{res} \circ \text{tr}(x) = \sum_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x$
Example: Constant $C_p$-Mackey functor $\mathbb{Z}$

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$\text{res} = \text{id}$

$\text{tr} = \times p$
Why Tambara Functors?

Recall: If $X$ is a $G$-spectrum then $\pi_k(X)$ is a Mackey functor.
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**Theorem (Brun, 2004)**
If $X$ is a commutative $G$-ring spectrum

Not Surprising
A Tambara functor = Mackey functor with extra structure.

$\mathcal{M}^G = \text{Category of } G$-Mackey functors
Symmetric monoidal product in $\mathcal{M}^G = \text{Box Product } \otimes$

Surprising
Commutative ring objects under $\otimes = \text{Commutative Green functors } \neq \text{ Tambara functors}$

Tambara functors = Commutative Green functors $++$

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Why Tambara Functors?

Recall: If $X$ is a $G$-spectrum then $\pi_k(X)$ is a Mackey functor.

**Theorem (Brun, 2004)**

If $X$ is a commutative $G$-ring spectrum then $\pi_0(X)$ is a $G$-Tambara functor.
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- $Mack_G = \text{Category of } G\text{-Mackey functors}$
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Commutative ring objects under $\Box$ $\neq$ Commutative Green functors

Tambara functors $\neq$ Commutative Green functors
A $C_p$-Tambara Functor, $S$

\[
\begin{array}{c}
\overset{\text{res}}{S(C_p/C_p)}
\end{array}
\quad \xrightarrow{\gamma} \quad
\begin{array}{c}
\overset{\text{tr}}{S(C_p/e)}
\end{array}
\]
A $C_p$-Tambara Functor, $S$

\[
\text{Comm } C_p\text{-Ring} \rightarrow \xrightarrow{\gamma} S(C_p/C_p) \xrightarrow{\text{res}} S(C_p/e) \xrightarrow{\text{tr}} \leftarrow \text{Comm } C_p\text{-Ring}
\]
A $C_p$-Tambara Functor, $S$

\[
\begin{array}{c}
\text{Comm } C_p\text{-Ring} \\
\text{Ring Homom}
\end{array} \rightarrow \quad \begin{array}{c}
\mathcal{S}(C_p/C_p) \\
\gamma \cdot \text{tr}
\end{array} \rightarrow \quad \begin{array}{c}
\mathcal{S}(C_p/e) \\
\text{res}
\end{array} \rightarrow \quad \begin{array}{c}
\text{Comm } C_p\text{-Ring}
\end{array}
\]

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A $C_p$-Tambara Functor, $S$

\[
\text{Comm } C_p\text{-Ring} \rightarrow \frac{S(C_p/C_p)}{\gamma} \rightarrow \text{Ring Homom} \rightarrow \frac{S(C_p/e)}{\gamma} \rightarrow \text{Comm } C_p\text{-Ring}
\]

Frobenius Reciprocity

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A Commutative $C_p$-Tambara GREEN Functor, $S$

Frobenius Reciprocity
A $C_p$-Tambara Functor, $S$

Comm $C_p$-Ring $\rightarrow$ $S(C_p/C_p)$

Ring Homom

$S(C_p/e)$ $\leftarrow$ Comm $C_p$-Ring

- Frobenius Reciprocity
- Norm = Multiplicative analogue of the transfer
A $C_p$-Tambara Functor, $S$

\[
\begin{array}{c}
\text{Comm } C_p\text{-Ring} \xrightarrow{\text{res}} S(C_p/C_p) \xrightarrow{N} S(C_p/e) \xrightarrow{\gamma} \text{Comm } C_p\text{-Ring} \\
\text{Ring Homom} \xrightarrow{\text{tr}} \end{array}
\]

- Frobenius Reciprocity
- $\text{Norm} = \text{Multiplicative analogue of the transfer}$
- $\text{res} \circ \text{tr}(x) = \sum_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x$

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Frobenius Reciprocity
Norm = Multiplicative analogue of the transfer

\[ \text{res} \circ \text{tr}(x) = \sum_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x \]
\[ \text{res} \circ N(x) = \prod_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x \]
A $C_p$-Tambara Functor, $S$

\[ \text{Comm } C_p\text{-Ring} \rightarrow S(C_p/C_p) \rightarrow S(C_p/e) \rightarrow \text{Comm } C_p\text{-Ring} \]

- Frobenius Reciprocity
- Norm = Multiplicative analogue of the transfer

\[ \text{res} \circ \text{tr}(x) = \sum_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x \]

\[ \text{res} \circ N(x) = \prod_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x \]

\[ N(a + b) = N(a) + N(b) + \text{tr}(-) \]
Frobenius Reciprocity

Norm = Multiplicative analogue of the transfer

\[ res \circ tr(x) = \sum_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x \]

\[ res \circ N(x) = \prod_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x \]

\[ N(a + b) = N(a) + N(b) + tr(-) \]

\[ \left[ N(tr(x)) = tr(-) \right] \]
Example: Constant $C_p$-Tambara functor $\mathbb{Z}$

$$\mathbb{Z} = C_p\text{-ring with trivial action}$$
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$N(x) = x^p$
Recall! Thm (Brun): $X$ a Comm. $G$-ring spectrum $\Rightarrow \pi_0(X)$ a $G$-Tambara functor.

Nice: To build an equivariant symmetric monoidal structure on $\text{Mack}_G$ under which Tambara functors are the equivariant commutative ring objects.
Recall!

**Thm (Brun):** $X$ a Comm. $G$-ring spectrum $\implies \pi_0(X)$ a $G$-Tambara functor.

- Ring objects under $\square$ are Green functors, *not* Tambara functors.
Recall!

- **Thm (Brun):** \( X \) a Comm. \( G \)-ring spectrum \( \implies \pi_0(X) \) a \( G \)-Tambara functor.
- Ring objects under \( \square \) are Green functors, *not* Tambara functors

Nice:

To build an *equivariant* symmetric monoidal structure on \( \text{Mack}_G \) under which Tambara functors are the *equivariant* commutative ring objects.
Definition (due to Hill and Hopkins)

A G-symmetric monoidal structure on $\mathcal{M}ack_G$ consists of a map

$$(-) \otimes (-): \mathcal{S}et^\text{Fin}_G \times \mathcal{M}ack_G \to \mathcal{M}ack_G$$

such that
A \textit{G-symmetric monoidal structure} on $\mathcal{M}ack_G$ consists of a map

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such that

1. $(X \amalg Y) \otimes M = (X \otimes M) \boxdot (Y \otimes M)$ and
2. $X \otimes (M \boxdot L) = (X \otimes M) \boxdot (X \otimes L)$
A \textit{G-symmetric monoidal structure} on $\text{Mack}_G$ consists of a map

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such that

- $(X \amalg Y) \otimes M = (X \otimes M) \boxtimes (Y \otimes M)$ and $X \otimes (M \boxtimes L) = (X \otimes M) \boxtimes (X \otimes L)$
- $X \otimes (Y \otimes M) = (X \times Y) \otimes M$. 

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Mackey and Tambara Functors

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**G-Symmetric Monoidal**

**Definition (due to Hill and Hopkins)**

A *G*-symmetric monoidal structure on $\text{Mack}_G$ consists of a map

$(-) \otimes (-): \mathcal{S}et^\text{Fin}_G \times \text{Mack}_G \to \text{Mack}_G$

such that

1. $(X \sqcup Y) \otimes M = (X \otimes M) \Box (Y \otimes M)$ and $X \otimes (M \Box L) = (X \otimes M) \Box (X \otimes L)$
2. $X \otimes (Y \otimes M) = (X \times Y) \otimes M$. 

Every Mackey functor $M$ defines a map $(-) \otimes M: \mathcal{S}et^\text{Fin}_G \times \text{Mack}_G \to \text{Mack}_G$.

A *G*-commutative monoid is a Mackey functor $M$ such that the map $(-) \otimes M$ extends to a functor.
**G-Symmetric Monoidal**

**Definition (due to Hill and Hopkins)**

A *G*-symmetric monoidal structure on $\text{Mack}_G$ consists of a map

$(-) \otimes (-) : \mathcal{I}et_G^{\text{Fin}} \times \text{Mack}_G \to \text{Mack}_G$

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Every Mackey functor $M$ defines a map $(-) \otimes M : \mathcal{I}et_G^{\text{Fin}} \to \text{Mack}_G$. 
Definition (due to Hill and Hopkins)

A $G$-symmetric monoidal structure on $\text{Mack}_G$ consists of a map

$(-) \otimes (-) : \text{Set}_G^{\text{Fin}} \times \text{Mack}_G \to \text{Mack}_G$

such that

1. $(X \sqcup Y) \otimes M = (X \otimes M) \square (Y \otimes M)$ and $X \otimes (M \square L) = (X \otimes M) \square (X \otimes L)$
2. $X \otimes (Y \otimes M) = (X \times Y) \otimes M$.

Every Mackey functor $M$ defines a map $(-) \otimes M : \text{Set}_G^{\text{Fin}} \to \text{Mack}_G$.

Definition (due to Hill and Hopkins)

A $G$-commutative monoid is a Mackey functor $M$ such that the map $(-) \otimes M$ extends to a functor.
From Now On: $G$ is a cyclic $p$-group.
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**Theorem (M)**

- constructed a computable $G$-symmetric monoidal structure on $\text{Mack}_G$
- a Mackey functor is a $G$-commutative monoid if and only if it has the structure of a Tambara functor.
From Now On: $G$ is a cyclic group of order $p^n$, $p$ prime

Theorem (M)
- constructed a **computable** $G$-symmetric monoidal structure on $\text{Mack}_G$
- a Mackey functor is a $G$-commutative monoid if and only if it has the structure of a Tambara functor.

Advantages:
- Concrete and computationally accessible
- Can describe this structure using diagrams
Building the $G$-symmetric monoidal structure

Step 1

For all subgroups $H$ of $G$, built symmetric monoidal functors

$$N^G_H : \text{Mack}_H \to \text{Mack}_G.$$
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$$N^G_H : \text{Mack}_H \to \text{Mack}_G.$$
Building the $G$-symmetric monoidal structure

**Step 1**

For all subgroups $H$ of $G$ **built** symmetric monoidal functors

$$N_H^G : \text{Mack}_H \rightarrow \text{Mack}_G.$$ 

$N_H^G M \leftrightarrow \text{“universal home for norms”}$
Building the $G$-symmetric monoidal structure

Step 1
For all subgroups $H$ of $G$ built symmetric monoidal functors $N^G_H : \text{Mack}_H \to \text{Mack}_G$.

- $N^G_H M \leftrightarrow \text{"universal home for norms"}$

Step 2
Defined the $G$-symmetric monoidal structure $(-) \otimes (-)$ by

\[
G/H \otimes M = N^G_H i^* H M (X \gg Y) \otimes M = (X \otimes M) \otimes (Y \otimes M)
\]
Building the $G$-symmetric monoidal structure

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For all subgroups $H$ of $G$ built symmetric monoidal functors

$$N_H^G : \text{Mack}_H \rightarrow \text{Mack}_G.$$ 

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Step 2
Defined the $G$-symmetric monoidal structure $(\cdot) \otimes (\cdot)$ by

- $G/H \otimes M = N_H^G i_H^* M$
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### Step 1
For all subgroups $H$ of $G$ built symmetric monoidal functors

$$N^G_H : \text{Mack}_H \to \text{Mack}_G.$$  

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### Step 2
Defined the $G$-symmetric monoidal structure $(-) \otimes (-)$ by

- $G/H \otimes M = N^G_H i^*_H M$
- $(X \amalg Y) \otimes M = (X \otimes M) \Box (Y \otimes M)$

### Step 3
Proved that $S$ is a Tambara functor if and only if $(-) \otimes S$ extends to a functor.

- $S = \text{Tambara functor} \iff G/K \to G/H \leadsto G/K \otimes S \to G/H \otimes S$
Building the $G$-symmetric monoidal structure

**Step 1**
For all subgroups $H$ of $G$ built symmetric monoidal functors

\[ N^G_H : \text{Mack}_H \rightarrow \text{Mack}_G \]

- $N^G_H M \leftrightarrow \text{“universal home for norms”}$

**Step 2**
Defined the $G$-symmetric monoidal structure $(-) \otimes (-)$ by

- $G/H \otimes M = N^G_H i^*_H M$
- $(X \boxtimes Y) \otimes M = (X \otimes M) \square (Y \otimes M)$

**Step 3**
Proved that $S$ is a Tambara functor if and only if $(-) \otimes S$ extends to a functor.

- $S = \text{Tambara functor} \iff G/K \rightarrow G/H \rightsquigarrow G/K \otimes S \rightarrow G/H \otimes S$
Let $M$ be a module.
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$$(N_{e}^{C_2} M)(C_2/C_2) = \left[ \mathbb{Z}\{M\} \oplus \frac{(M \otimes M)/C_2}{TR} \right]$$

$$(N_{e}^{C_2} M)(C_2/e) = M \otimes M$$
An Example: $N^{C_2}_e \mathbb{Z}/2$
An Example: $N_e^{C_2} \mathbb{Z}/2$

$$\left[ \mathbb{Z}\{[0],[1]\} \oplus (\mathbb{Z}/2 \otimes \mathbb{Z}/2)/C_2 \right] / TR$$
An Example: $N_e^{C_2} \mathbb{Z}/2$

\[
\left[ \mathbb{Z}\{[0], [1]\} \oplus \left( \mathbb{Z}/2 \otimes \mathbb{Z}/2 \right)/C_2 \right] / TR
\]

Simplifies to

\[
\mathbb{Z}/4 \times 2
\]

\[
\mathbb{Z}/2
triv.
\]
Thank You!