Additional Structure on the Category of Mackey Functors

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Overview

Goal (Broadest Sense)

- Understand the underlying algebra of Mackey functors and Tambara functors.
- Develop a concrete structure on the category of G-Mackey functors that provides a nice characterization of G-Tambara functors.

$G =$ cyclic group of order p^n (p prime)

Why Mackey Functors?

Let G be a finite abelian group. Let X be a G-spectrum.

k^{th} Stable Homotopy Group of X, $\underline{\pi}_k(X)$, $k \in \mathbb{Z}$

- For all $H \leq G$ define
$$\pi_k^H(X) = [\underline{S}^k, X]^H$$
- The collection of all $\pi_k^H(x)$ is a *Mackey functor*
- Define $\underline{\pi}_k(X)$ to be this Mackey functor.

Definition (due to Dress)

A *G-Mackey functor* $\underline{M}$ consists of a pair of functors

$$(M_*, M^*) : \mathscr{S}et_G^{Fin} \to \mathscr{A}b$$

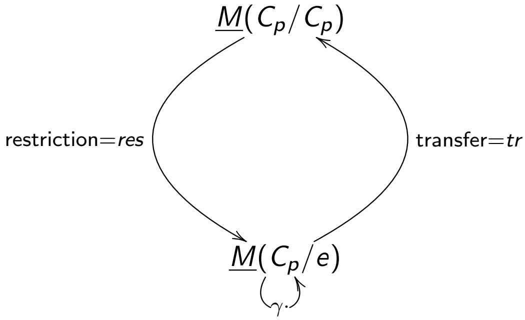
such that

- $M^*(X) = M_*(X) := \underline{M}(X)$ for all X in $\mathscr{S}et_G^{Fin}$
- $\underline{M}(X \amalg Y) = \underline{M}(X) \oplus \underline{M}(Y)$
- If the diagram below is a pullback diagram in $\mathscr{S}et_G^{Fin}$ then $M_*(h) \circ M^*(f) = M^*(g) \circ M_*(k)$ in $\mathscr{A}b$.

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{k} & W \end{array}$$

A C_p-Mackey Functor $\underline{M}$

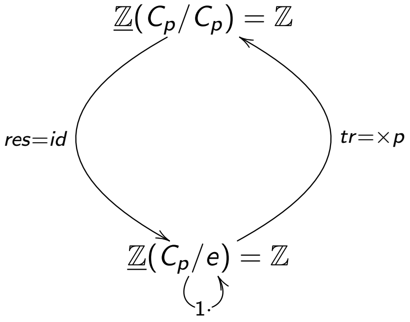
C_p - Cyclic group of prime order



- γ generates the Weyl group, $W_{C_p}(e)$
- $res \circ tr(x) = \sum_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x$

Example: Constant C_p-Mackey functor $\underline{\mathbb{Z}}$

$\mathbb{Z} = C_p$-module with trivial action



Why Tambara Functors?

Recall: If X is a G-spectrum then $\underline{\pi}_k(X)$ is a Mackey functor.

Theorem (Brun, 2004)

If X is a commutative G-ring spectrum then $\underline{\pi}_0(X)$ is a *G-Tambara functor*

Not Surprising

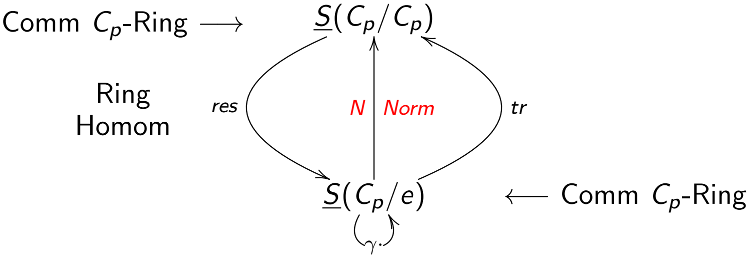
A Tambara functor = Mackey functor with extra structure.

- $\mathcal{M}ack_G$ = Category of G-Mackey functors
- Symmetric monoidal product in $\mathcal{M}ack_G$ = *Box Product* $\square$

Surprising

Commutative ring objects under $\square$	$=$	Commutative Green functors
	$\neq$	Tambara functors
Tambara functors	$=$	Commutative Green functors ++

A C_p-Tambara Functor, $\underline{S}$



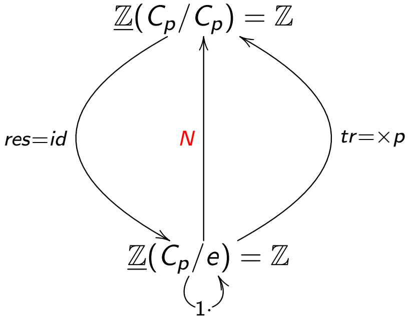
- Frobenius Reciprocity
- Norm = Multiplicative analogue of the transfer

- $res \circ tr(x) = \sum_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x$
- $res \circ N(x) = \prod_{\gamma^r \in W_{C_p}(e)} \gamma^r \cdot x$

- $N(a+b) = N(a) + N(b) + tr(-)$
- $\Big[N(tr(x)) = tr(-) \Big]$

Example: Constant C_p-Tambara functor $\underline{\mathbb{Z}}$

$\mathbb{Z} = C_p$-ring with trivial action



$N(x) = x^p$

An Equivariant Symmetric Monoidal Structure

Recall!

- **Thm (Brun):** X a Comm. G-ring spectrum $\Longrightarrow$ $\underline{\pi}_0(X)$ a G-Tambara functor.
- Ring objects under $\Box$ are Green functors, *not* Tambara functors

Nice:

To build an *equivariant* symmetric monoidal structure on $\mathcal{M}ack_G$ under which Tambara functors are the *equivariant* commutative ring objects.

G-Symmetric Monoidal

Definition (due to Hill and Hopkins)

A *G-symmetric monoidal structure* on $\mathcal{M}ack_G$ consists of a **map**

$$(-) \otimes (-) : \mathscr{S}et_G^{Fin} \times \mathcal{M}ack_G \to \mathcal{M}ack_G$$

such that

- $(X \amalg Y) \otimes \underline{M} = (X \otimes \underline{M}) \,\square\, (Y \otimes \underline{M})$ and
  $X \otimes (\underline{M} \,\square\, \underline{L}) = (X \otimes \underline{M}) \,\square\, (X \otimes \underline{L})$
- $X \otimes (Y \otimes \underline{M}) = (X \times Y) \otimes \underline{M}$.

Every Mackey functor $\underline{M}$ defines a map $(-) \otimes \underline{M} : \mathscr{S}et_G^{Fin} \to \mathcal{M}ack_G$.

Definition (due to Hill and Hopkins)

A *G-commutative monoid* is a Mackey functor $\underline{M}$ such that the map $(-) \otimes \underline{M}$ extends to a functor.

My Project!

From Now On: G **is a cyclic group of order** p^n, p **prime**

Theorem (M)

- constructed a **computable** G-symmetric monoidal structure on $\mathcal{M}ack_G$
- a Mackey functor is a G-commutative monoid if and only if it has the structure of a Tambara functor.

Advantages:

- Concrete and computationally accessible
- Can describe this structure using diagrams

Building the G-symmetric monoidal structure

Step 1

For all subgroups H of G **built** symmetric monoidal functors

$$N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G.$$

- $N_H^G \underline{M} \leftrightsquigarrow$ "universal home for norms"

Step 2

Defined the G-symmetric monoidal structure $(-) \otimes (-)$ by

- $G/H \otimes \underline{M} = N_H^G i_H^* \underline{M}$
- $(X \amalg Y) \otimes \underline{M} = (X \otimes \underline{M}) \Box (Y \otimes \underline{M})$

Step 3

Proved that $\underline{S}$ is a Tambara functor if and only if $(-) \otimes \underline{S}$ extends to a functor.

- $\underline{S}$ = Tambara functor $\iff$ $G/K \to G/H \rightsquigarrow G/K \otimes \underline{S} \to G/H \otimes \underline{S}$

Building the G-symmetric monoidal structure

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$$\mathbf{N_H^G} : \mathcal{M}\mathbf{ack_H} \to \mathcal{M}\mathbf{ack_G}$$

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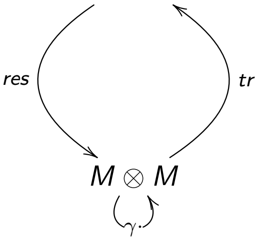
The C_2-Mackey Functor $N_e^{C_2} M$

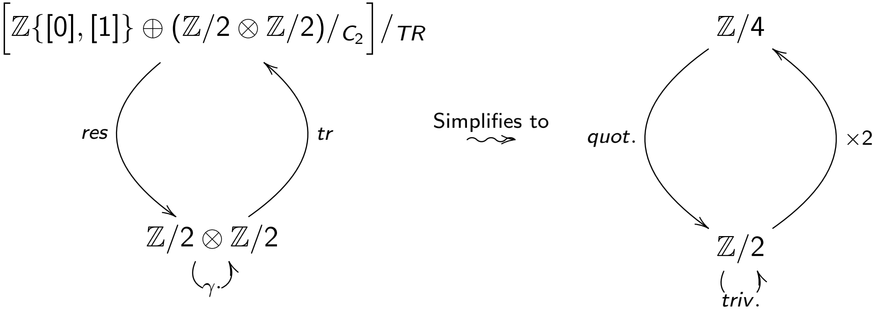
Let M be a module.

$$(N_e^{C_2} M)(C_2/C_2) = \Big[\overbrace{\mathbb{Z}\{M\}}^{\text{home for norms}} \oplus \overbrace{(M \otimes M)/_{C_2}}^{\text{Image of transfer}} \Big] /_{TR}$$



$$(N_e^{C_2} M)(C_2/e) = M \otimes M$$

An Example: $N_e^{C_2}\mathbb{Z}/2$



Thank You!