LVAIC Mathematics Contest – November 2, 2014

Do as many of these problems as you can.
No calculators of any sort, notes, or reference materials are allowed.
Your solutions must be complete and your work justified to receive full credit.
Each solution must be written on a separate piece of paper with your team's code on it.

1. 2014 lines are drawn in the plane, with no two lines parallel and no three lines meeting at a point. Half of the lines are colored orange and half are blue. Call a point of intersection between the two lines a *good* point if both of the lines are the same color, and call it *bad* otherwise. Are there more good points or bad points? How many more?

2. Find all positive integer solutions to the equation: $abc - 2 = a + b + c$

3. Since we are just a couple days after Halloween, you decide to go to a place inhabited by vampires. In this land, everyone is either a human or a vampire. Humans always say what they believe to be the truth, while vampires always say what they believe to be a lie. Unfortunately, the people who live here are also sane or insane. Sane individuals correctly know what is true and what is false. Insane individuals believe that what is false is actually true, and they believe that what is true is actually false.

   You meet two brothers who give the following statements:

   Bob: I am human.
   Joe: I am human.
   Bob: My brother is sane.

   What can you conclude regarding the sanity of the two individuals and whether each of them is human or a vampire?

4. The numbers from 1 to 2014 are placed in a 2 x 1007 array, and then the ends are joined to make a bracelet. Two boxes are neighbors if they share an edge or a corner. Each box in the array then has five neighbors: the box immediately above or below, the two boxes to the left and the two boxes to the right. For every pair of neighbors x and y, compute $|x - y|$ and call the *weight* of the bracelet the largest of all possible neighborly differences.
   (a) What is the largest possible weight of a bracelet?
   (b) What is the smallest possible weight of a bracelet?

5. Compute the limit

$$\lim_{n \to \infty} \sqrt[n]{\frac{1}{4} \cdot \frac{3}{8} \cdots \frac{2n-1}{4n}}$$

6. Suppose there exists an isosceles right triangle that can be completely covered by 2014 circles that each have a radius of 2 inches. Can this same triangle be completely covered by 8056 circles that each have a radius of 1 inch? If so, explain why. If not, explain why not.

7. You are very bored and decide to play a game. You will roll a die and count the number of pips (spots) on the top face of the die. You roll the die again and add the number of pips on the top face of the die from this roll to the number from the previous roll. You keep rolling the die and adding the number of pips, creating a running total, until your total is more than 2014. What number or numbers are most likely to be your final total?

8. You are very bored and decide to play a game. You cut out 2016 circles and place them in 6 piles of 336 circles each. (Yes, I know it's not 2014, but 6 is not a factor of 2014.) You number the circles in each pile from 1 to 336. You then paint the 6 piles the colors red, orange, yellow, green, blue, and purple respectively. Next, you shuffle the 2016 circles.

   (a) You randomly draw 2 of the 2016 circles from the pile. What is the probability that both of the circles have the number 100 on them?

   (b) You randomly draw 2 of the 2016 circles from the pile. Your friend looks at your choices before you can see them and tells you that at least one of the circles is blue. What is the probability that both of the circles have the number 100 on them?

9. Find a closed form solution to $\sum_{n=1}^{N} n \cdot n!$

10. An equilateral triangle is inscribed in a circle of radius 1. One of the sides of the triangle is then removed. Find the radius of the largest circle that can be inscribed inside the circle and the remaining two sides of the triangle, as shown in the picture below.