# LVAIC Mathematics Contest – November 3, 2013

*Do as many of these problems as you can.*
*No calculators of any sort, notes, or reference materials are allowed.*
*Your solutions must be complete and your work justified to receive full credit.*
*Each solution must be written on a separate piece of paper with your team's code on it.*

1. Kevin is walking at a constant rate of 3 miles per hour down a street along which runs a streetcar line. He notices that, while 40 streetcars pass him traveling in the same direction as he walks, 60 pass him in the opposite direction. He does not understand why this happens, but thinks this is cool. Assuming that the streetcars are equally spaced and traveling at a constant speed, what is the speed of the streetcars?

2. In a flock of chickens, there is a pecking order. That is, for every two chickens A and B, either A always pecks B or B always pecks A. An Emperor chicken is a chicken which pecks all other chickens in the flock. Not every flock has an Emperor. A King chicken is a chicken A such that for every other chicken B, either A pecks B, or A pecks another chicken C such that C pecks B. Prove that there exists a King chicken in every flock of size at least 2.

3. If *a* is any integer, must there exist integers *x* and *y* such that $x^2 - y^2 = a^3$? If so, prove it. If not, prove why not.

4. Suppose that $b_1 = 1$ and that $b_{n+1} = \cos(\arctan(b_n))$ for all integers $n > 1$. Find $\lim_{n \to \infty} b_n$.

5. We are in Knod (somewhere southeast of Eugene, Oregon), the land of Knights and Knaves, where Knights only tell true statements and Knaves only make false statements. Every inhabitant of Knod is either a Knight or a Knave. We are trying to determine the location of Gary, who escaped from the Lehigh Valley over the summer and is somewhere in Knod. We enter the hamlet of Briar, and several of the inhabitants approach us. The following exchange takes place between them. (Their names are all correct and known to you.)

   *Hal: Gary was here today.*
   *Sal: Gary was not here today.*
   *Cal: Gary was here yesterday.*
   *Mal: Gary was not here today and Gary was not here yesterday.*
   *Val: Cal is a Knight or Mal is a Knave.*
   *Hal: Val is a Knave or Cal is a Knight.*

   Is it possible to determine if Gary was in Briar today? How about yesterday?

   Note that the statement "A or B" is true if A is true, or if B is true, or if both A and B are true. It is false only if both A is false and B is false. Also, the statement "A and B" is true only if both A is true and B is true. It is false otherwise.

6. Let *u*, *v*, and *w* be positive integers and assume the only positive integer that divides all three of them is 1. We also have:

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{w}$$

An example is 1/30 + 1/6 = 1/5. Notice that 30 + 6 = 36 is a perfect square. Prove this is always true. That is, prove that $u + v$ is always a perfect square.

7. You have 2013 = 3 * 11 * 61 small cubes numbered 1 through 2013. You place the numbered cubes in three 11 by 61 rectangular grids. Is it possible for the numbers of the cubes in each of these 11 by 61 grids to have the same sum?

8. Pat, Chris, and Sal decide to play a game of dodgeball. Since there are three of them, they can't split into two teams, so they decide on a new set of rules. They will draw straws to see the order in which they can throw the ball. Then, they will each throw the ball in this order provided they aren't hit before their turn arrives. Once they are hit, they are eliminated. They will continue to take turns until only one individual is left standing. Pat is excellent at throwing the ball and never misses the target. Chris only hits the target 80% of the time. Sal could use some practice, as Sal only hits the target 50% of the time. If each individual uses the strategy that maximizes their own chances of winning (and provided no errant balls accidentally hit one of them), find the probability of winning for each person.

9. Prove that the sequence 11, 111, 1111, 11111, … contains no numbers that are square numbers.

10. Trisect the four right angles of the unit square, creating the octagon in the center. What is the area of the octagon?