LVAIC Mathematics Contest – November 14, 2010

No calculators, cell phones or notes are allowed. Your solutions must be complete and your work justified to receive full credit. Write up each solution on a separate sheet of paper!

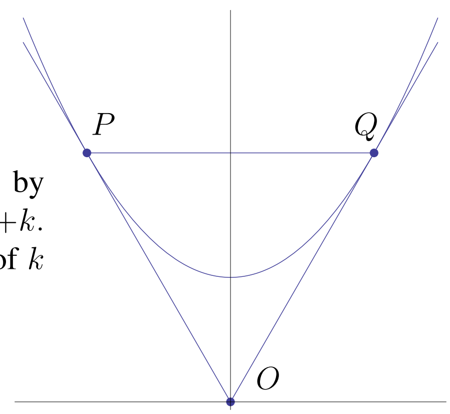
1. How many positive integers less than 2010 have exactly 9 factors? (For this problem, count both 1 and the number itself as factors: for instance, 6 has four factors: 1, 2, 3 and 6.)

2. The expression $1/2/3/4$ is ambiguous – we need to add parentheses to figure out what order to perform the divisions. For instance, we could get $(1/2)/(3/4) = 2/3$, but we could also get $(1/(2/3))/4 = 3/8$. Here's an expression:

$$1/2/3/4/5/6/7/8/9/10/11$$

   Can you make this expression an integer by appropriately inserting parentheses? If so, find a way to do it. If not, prove it's impossible to make the expression an integer.

3. You have a giant 2010×2010 grid, and you put the numbers from 1 to 2010 into the grid so that each number between 1 and 2010 appears exactly once in each row and also exactly once in each column. (This is called a *Latin square.*) Your friend is in a destructive mood, and she crosses out n rows and n columns from your grid. Then you notice that what remains is a Latin square with the remaining $2010 - n$ numbers. Show that n must be at least 1005.



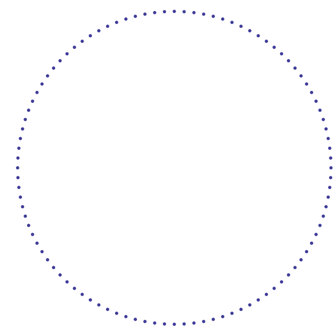
4. Let O be the point $(0, 0)$, and form a triangle $\triangle OPQ$ by drawing 2 tangent lines through O to the parabola $y = x^2 + k$. Find coordinates for the points P and Q and the value of k to make $\triangle OPQ$ equilateral.

5. Five pirates of different ages have a treasure of 100 gold coins. These pirates live by the pirate code, and that code says that all treasure must be split in the following way: The oldest pirate proposes how to share the coins, and all pirates remaining will vote for or against it. If 50% or more of the pirates vote for it, then the coins will be shared that way. Otherwise, the pirate proposing the scheme will be thrown overboard, and the process is repeated with the pirates that remain. Pirates love gold, but they also like throwing people overboard, so if a pirate will end up with the same number of coins by voting yes or by voting no, that pirate will vote no.

   Each pirate will act in a way to maximize the number of gold coins they receive, and they are very good at reasoning. How many gold coins will each of the pirates end up with?

6. Compute the indefinite integral: $\displaystyle\int \frac{xe^{3x}}{(3x+1)^2}\,dx$

7. Suppose 2010 points are equally spaced on a circle (the picture shows 100 - count them if you don't believe me). Hannah and Rebecca each choose 1005 of these points, and it's possible their two sets overlap - we do *not* assume the 2 sets are disjoint. Hannah must keep her set fixed, but Rebecca is allowed to rotate her entire set by any angle she wants.



   (a) Show that Rebecca can choose an angle so that her rotated set includes at least 503 points from Hannah's set.

   (b) Suppose there is a positive integer k so that *every* rotation of Rebecca's set includes k or more points from Hannah's set. What is the largest possible value of k?

8. Three sisters are identical triplets. The oldest (by minutes) is Sarah, and Sarah always tells the truth. The next oldest is Sue, and Sue always lies. Sally is the youngest of the three, and she sometimes lies and sometimes tells the truth. You meet the three sisters, who are sitting next to each other in a row. You want to figure out who's who, so you ask each one a question, as follows.

   First, you ask the sister who is sitting on the left, "Which sister is in the middle?" This sister answers, "Oh, that's Sarah." Then you ask the sister in the middle, "What is your name?" This sister says, "I'm Sally." Finally, you ask the sister on the right, "Which sister is in the middle?" This sister replies, "She is Sue."

   Can you figure out who's who? If so, explain how. If not, explain why it is impossible to do so.

9. Let S be the set of points (a, b) with $0 \leq a, b \leq 10$ and such that the system of equations

$$x + y + z = a \qquad \text{and} \qquad xy + yz + zx = b$$

   has at least one solution with all three of x, y, and z real. What is the area of S?

10. Let A be a set of 20 distinct integers, all between 1 and 100 (inclusive). Then $A + A$ is the set of all integers of the form $x + y$, with x and y both in A. (So if 1 and 3 are in A, then $4 = 1 + 3$ and $2 = 1 + 1$ are both in $A + A$.)

   (a) Show that there is a number r in $A + A$ which is a sum in at least two different ways. (For instance, if $2, 3$, and 4 are in A, then 6 is in $A + A$ in two different ways since $6 = 2 + 4$ and $6 = 3 + 3$. We don't care about order, however: $6 = 4 + 2$ and $6 = 2 + 4$ do not count as being different.)

   (b) Show that there are at least 39 distinct elements of $A + A$.