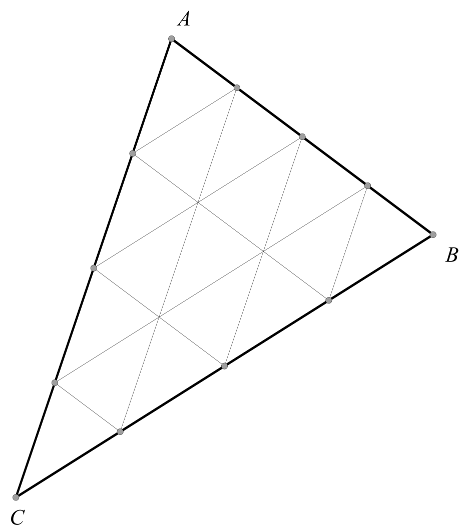
LVAIC Mathematics Contest – November 14, 2009

Do as many of these problems as you can.
No calculators, notes, or reference materials are allowed.
Your solutions must be complete and your work justified to receive full credit.
Write up each solution on a separate sheet of paper!

1. In $\triangle ABC$, every side is divided into n equal parts and from each division point, two lines are drawn parallel to two other sides. Count the number of parallelograms that can be seen inside $\triangle ABC$.



2. For a positive integer n, let $t(n)$ be the number of ways of writing n as a sum of one or more consecutive positive integers. For example, $t(15) = 4$ as $15 = 15$, $7 + 8$, $4 + 5 + 6$, or $1 + 2 + 3 + 4 + 5$. Find $t(2{,}009{,}000{,}000)$.

3. A dart is thrown at random at a square target. Find the probability that it lands as close or closer to the center as it does to any of the sides.

4. Suppose we have 11 consecutive integers and we multiply them together. What is the largest positive integer that you can guarantee divides this product?

5. Let n and k be positive integers with $k < n$. Suppose we arbitrarily choose a real number x from the interval $[0, 2n]$. What is the probability that $\lfloor (n+1)x \rfloor = \lfloor nx \rfloor + k$?

   Note that $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

6. Consider a circle of radius 1 centered at $(0, 0)$. We choose A to be the point $(0, 0)$, B to be the point $(1, 0)$, and C an arbitrary point in the interior of the circle. What is the probability that the triangle ABC contains an angle of more than $90°$?

7. Let $a_0, a_1, a_2, \ldots, a_9$ be a sequence of length 10 consisting of single digits. That is, each a_k is an integer between 0 and 9 inclusive. Note every digit must appear in the sequence, and some of the digits may appear more than once. This sequence has the strange property that for each k, a_k is the exact number of times that the digit k appears in the sequence. Thus if $a_3 = 2$, then the digit 3 appears exactly twice as a term in the sequence.

   (a) Find such a sequence.

   (b) Show that you answer is unique.

8. Evaluate the infinite product $\sqrt[2]{2}\sqrt[4]{4}\sqrt[8]{8}\,\sqrt[16]{16}\cdots$.

9. Find all real solutions to the pair of equations

$$\begin{aligned} x &= y(x-z) \\ x &= z(x-y). \end{aligned}$$

10. Suppose that you have two counters in a bag. You know that each counter is either white or black (with equal probability), but you don't know if they are both white, both black, or one of each. You draw one of the counters from the bag and notice that it is white before putting it back. You repeat this a total of 2009 times, each time noticing that the counter you draw happens to be white before placing it back in the bag.

    On the 2010$^{\text{th}}$ time you draw a token from the bag, what is the probability that it will again be white?