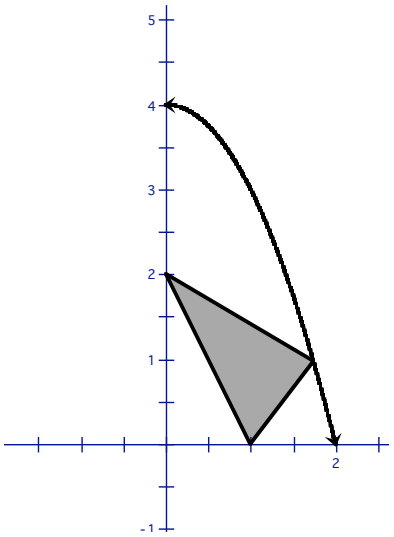
LVAIC Mathematics Contest – November 1, 2008

Do as many of these problems as you can.
No calculators or notes are allowed.
Your solutions must be complete and your work justified to receive full credit.
Write up each solution on a separate sheet of paper.

1. A triangle is formed by joining points $(1, 0)$, $(0, 2)$ and P, where P is a point in the first quadrant on the graph of $y = 4 - x^2$. What is the largest possible area for the triangle (see the picture).



2. Define the sequence x_n recursively by the equations

$$x_1 = 1, \quad x_n = 2x_{n-1} + (-1)^n.$$

What is x_{2008}?

3. Welcome to the LVAIC Insane Asylum. There are exactly two types of inhabitants of the asylum, doctors and patients. Each inhabitant is also sane or insane. Sane inhabitants are totally accurate they believe all true statements and no false ones. Insane inhabitants are totally deluded they believe all false statements and no true ones. Your job is to find and remove any sane patient or insane doctor. You discover that occupant A believes the statement that occupant B is insane and that occupant B believes the statement that occupant A is a doctor. What should you do?

4. Starting with the lines $y = x$ and $y = 0$, consider the set S of midpoints of line segments of length 1 that have one endpoint on the line $y = x$ and one endpoint on the line $y = 0$.

The points of S form a closed curve. Find a formula describing that curve.

5. What is the area of the region in the plane whose points (x, y) satisfy the inequality

$$|x| + |y| + |x + y| \leq 2?$$

6. Liz and Gary decided that they needed to buy some stamps. Liz sent Gary to the post office to buy the stamps. Knowing Gary's habits, she gave him only enough money to purchase exactly the stamps that they needed. Gary bought some 1 cent stamps, and three quarters as many 2 cent stamps as 1 cent stamps. He bought three quarters as many 5 cent stamps as 2 cent stamps, and he bought exactly five 8 cent stamps. He paid for all of these stamps with a single bill and there was no change.
How many stamps of each type did he buy?

   Note: Bills come in denominations of \$1, \$2, \$5, \$10, \$20, \$50, \$100, \$1000, and \$10,000. Show that your solution is unique.

7. Suppose we have six consecutive integers and we multiply them together. What is the largest positive integer that you can guarantee divides this product?

8. Let $f(x) = x^4 + x^3 + x^2 + x + 1$. Find the remainder when $f(x^5)$ is divided by $f(x)$.

9. Evaluate the integral $\displaystyle\int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\, dx$.

10. You have exactly 2008 pennies on a table. (This is all that remains of your retirement investments.) You are allowed to choose any subset of 1004 of the pennies and flip them over, and you can do this as many times as you like. What is the maximum number of heads you can get using this process, regardless of the initial configuration of the pennies?