LVAIC Mathematics Contest – November 3, 2007

Do as many of these problems as you can.
No calculators or notes are allowed.
Your solutions must be complete and your work justified to receive full credit.
Write up each solution on a separate sheet of paper!

1. A bug who is initially positioned at the point $(0,0)$ is trapped between the horizontal lines $y=0$ and $y=2$. Each step the bug takes moves it over one unit to the right in the x direction and either one step up or one step down in the y direction. For instance, the bug might begin a walk by traveling along the path

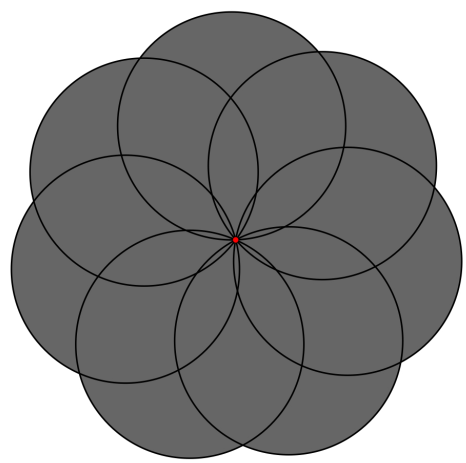
$$(0,0) \to (1,1) \to (2,0) \to (3,1) \to (4,2) \to (5,1) \to \cdots$$

Where will the bug be after exactly 2007 steps? How many different 2007 step paths are there?

2. Find the last digit of $2007^{\left(2007^{2007}\right)}$.

3. Find all positive integers x and y such that $7^x - 3^y = 4$.

4. Suppose that n discs intersect symmetrically about a single point (as shown in the figure at the right, with $n=7$). What is the total area covered by the discs?



5. Let S be a collection of 2007 points in the plane. Call a point P a *halving* point if there is a line l through P with exactly 1003 points of S on each side of l.

   (a) Show that any set of 2007 points must have at least one halving point.

   (b) What are the maximum and minimum number of halving points possible? Give examples for each possibilty.

6. Let p and q be consecutive prime numbers (i.e., there are no primes between them) with $3 < p < q$. Show that $p+q$ has at least three factors r, s and t, with $1 < r < s < t < p+q$.

7. A calculator is broken so that the only keys that still work are the $\sin$, $\cos$, $\tan$, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. On the other hand, the computations are infinitely accurate. All functions are using radians.

   The display initially shows 0. Show how to press a sequence of these 6 buttons to get the value $3/4$ to appear on the display.

8. Several identical square sheets of paper lie on a rectangular table so that their sides are parallel to the edges of the table (the sheets may overlap with one another). Prove that it is possible to stick several pins in the table so that each sheet is fixed to the table by exactly one pin.

9. Find the value of the integral:
$$\int_0^{\pi/2} \ln(\sin x) dx$$

10. There are two identical twins, one always lies and one always tells the truth. One of them is named John, but we don't know if that's the liar or the truth teller. You can ask one of them exactly one question which that brother can answer yes or no. Find out which brother is John. Now do it with a question that is at most three words long (and is a legitimate English sentence).

11. Find all possible values for positive integers a and b so that the polynomial x^2+x+1 divides $x^a + x^b + 1$.