LVAIC Mathematics Contest – October 21, 2006

Do as many of these problems as you can.
No calculators or notes are allowed.
Your solutions must be complete and your work justified to receive full credit.
Write up each solution on a separate sheet of paper!

1. There is a pile of 2006 stones, and two players (we'll call them Isaac and Gottfried) take turns removing either 3 stones or 5 stones, with Isaac playing first. The game ends when a person can't go (for example, if there are only 2 stones remaining), and that person loses the game.

   Is there a winning strategy for either player? If so, who wins and what is a winning strategy?

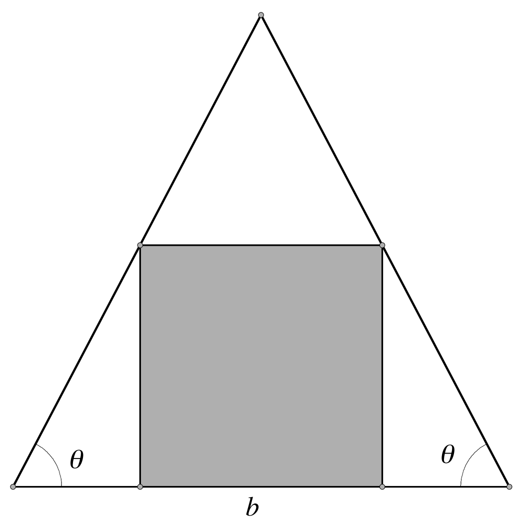
2. Let $x_0 = 1$, and $x_1 = 2$. For each integer $i > 1$, let $x_i = \dfrac{x_{i-1} + x_{i-2}}{2}$.

   Compute $\lim_{i\to\infty} x_i$.

3. The number 2006! has a lot of zeroes at the end. How many?

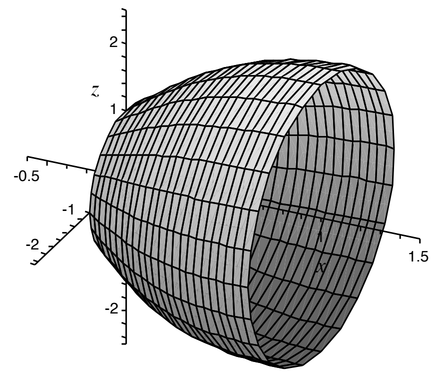
4. An isosceles triangle with base length b and base angle θ has an inscribed square, as shown below.

   Express the area of the square in terms of b and θ.



5. Let's list the numbers 1 to 2006 in an unusual order. Start by listing 1. Then list all the primes in numeric order. Then list all products of two primes, including squares, in numeric order (4, 6, 9, 10, etc.), then all products of 3 primes (8, 12, etc.), then all products of 4 primes and so on.

   What will be the last number listed?



6. Let $f(x) = \dfrac{2x}{x^2+1} + k$ be defined over the interval $0 \le x \le 1$, where k is a real constant. A solid of revolution may be formed by rotating this function about the x-axis. Find the value of k that will minimize the volume of the resulting solid.

7. An infinite grid is designed with the following rule: entry a_{ij} will be the smallest non-negative integer that does not appear either in the column above a_{ij} or in the row to the left of a_{ij}. The beginning of the array is given here. Note, for example, that the fourth entry on the sixth row is 6 because 0, 1, 2, 3 appear above the entry, while 4, 5 appear to the left of the entry.

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 0 & 3 & 2 & 5 & 4 & \cdots \\ 2 & 3 & 0 & 1 & 6 & 7 & \cdots \\ 3 & 2 & 1 & 0 & 7 & 6 & \cdots \\ 4 & 5 & 6 & 7 & 0 & 1 & \cdots \\ 5 & 4 & 7 & 6 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

   What number will appear in the 8^{th} column of the 2006^{th} row?

8. Suppose two disjoint circles with different radii are placed in the plane. Draw the two rays from the center of circle 1 tangent to circle 2, and let P and Q be the points where the rays intersect circle 1. Repeat the process from the center of circle 2 to form points R and S.

   Prove that P, Q, R, and S are the vertices of a rectangle.