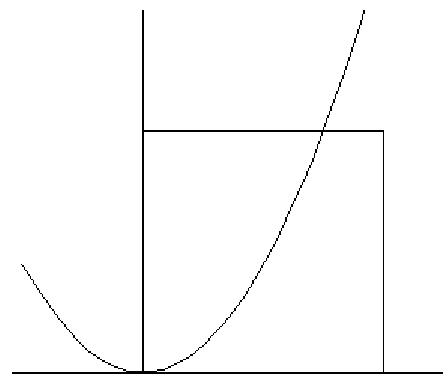
LVAIC Mathematics Contest – Oct. 22, 2005

Do as many of these problems as you can. No calculators or notes are allowed. Your solutions must be complete and your work justified to receive full credit. Write up each solution on a separate sheet of paper!

1. In honor of the 16th annual LVAIC Mathematics Competition: Take a fraction $\frac{a}{b}$ in lowest terms, where a and b are positive integers, and then compute the product ab. Find the number of different fractions that will give a product $ab = 16!$ (Recall: $16! = 16 \times 15 \times 14 \times \ldots \times 1$)

2. Find the value of k so that the graph of the parabola $y = kx^2$ cuts the unit square (with corners (0, 0), (0, 1), (1, 0) and (1, 1)) into two regions with the same area, as in the picture.



3. Rebecca and Hannah are playing a game. Their goal is to write down each number from 1 to 2005. They take turns as follows:
   - Rebecca writes the number 1
   - Hannah then writes down all the prime numbers in increasing order: 2, 3, 5, 7, 11, …
   - Next, Rebecca writes down 2 times all the primes, also in increasing order: 4, 6, 10, 14, 22, …
   - Next, Hannah writes down 3 times all the primes (skipping a number if it has already appeared on the list): 9, 15, 21, 33, …
   - Now Rebecca writes down 4 times all the primes: 8, 12, 20, 28, …

   The game continues like this until all of the numbers are written down (with the understanding that a number is never written down more than once). If there are no numbers to write down during someone's turn, their turn is skipped.

   a. Who writes down the number 2005?

   b. What is the last number written down? Who writes this number down?

4. Find, with proof, all positive integers n so that n^n is divisible by $n!$

5. Suppose x and y are positive real numbers. What is the smallest possible value of $xy + \frac{y}{x} + \frac{x}{y} + \frac{1}{xy}$? Prove your answer.

[More fun/torture on the other side!]

6. Suppose $f(x)$ is a continuous function with a continuous inverse function $f^{-1}(x)$.

Find all such functions satisfying $\displaystyle\int_0^{f^{-1}(x)} f(t)\,dt = x^2$

7. Here is a game involving two players and a pile of integers from 1 to 10 in the middle of a table. Player A removes one of the integers from the pile, then player B removes one, and so on. The players alternate turns, each one removing an integer from the pile until the product of the integers that remain in the pile is no longer divisible by 6. When this happens, the game ends and the last player to move is the winner. Which player has a winning strategy, and what is that strategy?

8. Rolling averages: In your spare time, you roll an ordinary six-sided die repeatedly and record your rolls as a sequence a_1, a_2, a_3, … (where each a_n is an integer from 1 to 6). After doing this for an infinite number of rolls, you notice something curious about your sequence: The number a_n is always the average of the two numbers a_{n-1} and a_{n+1} (for all $n > 1$). Must all of the a_n be the same number? If so, prove it – if not, give a specific counterexample.

9. A lattice point in the *xy*-plane is simply a point whose coordinates are both integers. For each lattice point (m, n) with $m > 0$ and $n > 0$, draw a circle centered at (m, n) that goes through the origin. [You should have an infinite number of circles; go ahead and draw them – we'll wait.] Show that, for each integer $k \geq 0$, there is some lattice point (m, n) that is on exactly k of these circles.

10. A checkerboard problem: A *liz* is a special type of piece in chess that only attacks squares that are horizontally or vertically adjacent to it.

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	**	*liz*	**		
		**			

a) What is the minimum number of *lizzes* that can be placed on a 4×4 checkerboard so that every square is either occupied by a *liz* or attacked by one? Prove your answer is optimal.

b) Same question for a 5×5 board.