LVAIC Mathematics Contest – Oct. 30, 2004

Do as many of these problems as you can. No calculators or notes are allowed. Your solutions must be complete and your work justified to receive full credit. Write up each solution on a separate sheet of paper!

1. Which number is bigger: 2004^{4002} or 4002^{2004}?

2. Find all positive values of n which make both $2^n - 1$ and $2^n + 1$ both prime numbers (1 is NOT a prime number!).

3. The numbers from 1 to 2004 are placed in a 6 x 334 grid.

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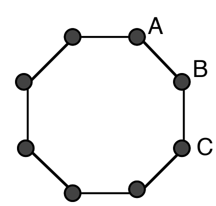
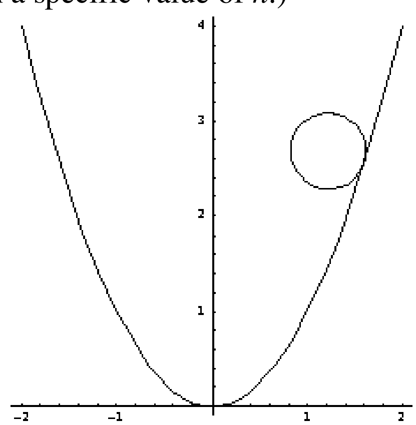
   a) Is it possible to place the numbers in such a way that the 334 column sums are all identical? If so, show how and find the common column sum. If not, show why such an arrangement is impossible.
   b) Is it possible for half of the column sums to be even numbers and half be odd numbers? If so, show how. If not, again explain why it's not possible.

4. Suppose there are n people stationed at the vertices of a regular n-gon, with one person at each vertex. Each person has some paint, and together, they have exactly as much paint as they need to paint the entire perimeter of the n-gon. Suppose person A begins painting, moving clockwise around the perimeter. If A has enough paint, then A can reach person B, waiting at the next vertex, and transfer her remaining paint to B. Then B can continue the process. Show that regardless of how the paint is initially distributed, there is always some person who can start the painting so that the entire perimeter will be painted. (Your argument should not depend on a specific value of n.)



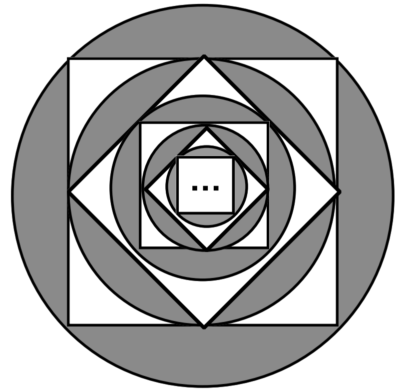
5. A circle rolls down the parabola $y = x^2$ so that it is always tangent to the parabola, as in the picture. What is the radius of the largest circle that will not get stuck at or before it rolls past the origin?



[See other side for more fun & games.]

6. Let $a_1 = 1$ and $a_n = \begin{cases} 2a_{n-1} & \text{if } a_{n-1} \text{ is odd} \\ a_{n-1}+1 & \text{otherwise.} \end{cases}$ for $n > 1$. Find a_{2004}.

7. A circle of radius 1 has a square inscribed, then that square has a circle inscribed, and that circle has a square inscribed, and so on, ad infinitum, as in the picture. Find the shaded area.



8. A two-person game is played with the numbers from 1 to 100 as follows:

   a. Liz picks any number from 1 to 100 (inclusive) and gives it to Gary.
   b. Gary then picks a different number between 1 and 100 and he either adds it to Liz's number or multiplies it by Liz's number. He then passes the result back to Liz. (Thus, Gary chooses a number and an operation, either + or x.)
   c. Liz then does the same thing Gary just did – she picks a number between 1 and 100 (different from all of the numbers selected so far) and either adds it to or multiplies it by what Gary passed to her. Then she passes the new result back to Gary.

   The 2 players continue in this way until all of the numbers from 1 to 100 have been exhausted, each time picking a number that has not been chosen yet and either adding it to or multiplying it by the number that was passed to them.

   Liz wins the game if the final result is even: Gary wins if it's odd. Can either player devise a strategy so that they will always win? If so, who? Describe such a strategy or show that neither player has a winning strategy.

9. Pedro throws two different types of pitches, a fastball (F) and a curve (C). His manager has told Pedro that he may never throw the curve three times in a row. How many different ways can Pedro throw ten pitches and keep his manager happy? For example, FFCFCCFFFF is fine, but FCCCCFCFFF is not.

10. Two circles are placed in a square of side length 1 so that the circles are tangent to each other and at least 2 sides of the square. The centers of the circles are on a diagonal of the square, as in the picture. What is the maximum area the 2 circles can cover?