LVAIC Mathematics Contest - Nov. 1, 2003

Do as many of these problems as you can. No calculators or notes are allowed. Your solutions must be complete and your work justified to receive full credit. Write up each solution on a separate sheet of paper!

1. An ant is initially at 0 on a number line. The ant crawls around from number to number, and it can only move one step at a time. First, the ant moves one step to the right, to the number 1. Then it moves left 2 steps, reaching the number –1 on its 3rd step. Then it reverses direction again, going right for the next 3 steps (reaching the number 2 on its 6th step). Then it goes left 4 steps, and so on, alternating left and right blocks of increasing length. How many steps will the ant have taken when it first reaches the number 2003?

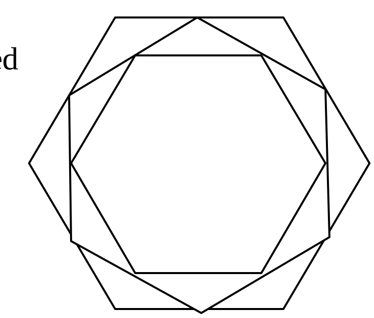
…	–3	–2	–1	0	1	2	3	…

2. Suppose $f(x)$ is a differentiable function defined for all real numbers with $f(x) > x$ for all x ? a, and with $f(a) = a$ (where a is some given constant). Show that $f'(a) = 1$.

3. Here's a 2-player game: The numbers from 1 to 90 are placed in a box. Player A removes a number from the box, then B removes a number, then A, and so on, with the payers alternately removing the numbers from the box. After all 90 numbers have been chosen, player A wins if the sum of all the numbers she has selected is even, while player B wins if the sum of the numbers player A selected is odd. Assuming both players follow an optimal strategy, who will win? What is the optimal strategy?

4. 100 ping pong balls, numbered 1 – 100, are placed in a box. You can remove two balls from the box and replace them with a new ball whose number is the sum of the numbers of the 2 balls you removed. This process is repeated many times; each time 2 balls are removed from the box and replaced by a ball with their sum. (During this process, it is possible for the box to have several balls with the same number.) After this process has been going on for awhile, you notice there are only two balls left in the box. The number on one of these balls is 2003. What is the number on the other ball?

5. A regular hexagon A has the midpoints of its edges joined to form a smaller hexagon B, and then this process is repeated by joining the midpoints of the edges of B to get a third hexagon C. What is the ratio of the area of C to the area of A?



More problem solving pleasure awaits you on the other side!

6. Suppose a b and c are complex numbers with

$$a + b + c = 2 \qquad a^2 + b^2 + c^2 = 3 \qquad abc = 4.$$

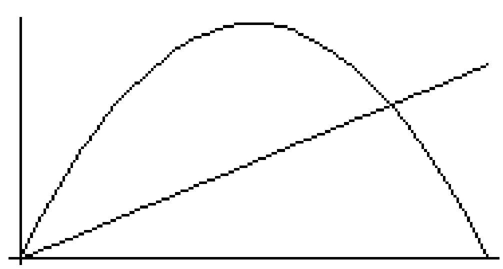
Find $a^3 + b^3 + c^3$. (It is not necessary to find a, b and c to do this!)

7. A 2-person game is played on a 4 x 4 grid as follows. Player A places the number 1 in some square of the grid, then player B places the number 2 in a square next to the square containing the 1. (‘Next to’ means above, below, left or right, but NOT diagonal.) Then player A places the number 3 next to the number 2, player B places the number 4 next to the 3, and so on. (The grid is shown after four moves have been made in a sample game below.) If all 16 numbers are placed in the grid legally, then the game is a draw; otherwise, a player loses if he cannot move. If both players follow optimal strategies, which player (if any) will win? Describe their optimal strategy.

	1	**2**	
		3	**4**

8. There is a positive even number N that cannot be written as the sum of two odd, composite numbers, but every number larger than N can be written as such a sum. (As an example, we can write 48 = 27 + 21, but there is no way to write 6 as the sum of two odd, composite numbers.) Find the value of N, and show that all larger even numbers can be expressed as the sum of two odd, composite numbers. (A number r is *composite* if $r = st$ with $s > 1$ and $t > 1$.)

9. Let R be the region below the graph of $y = x - x^2$ that’s above the x-axis. Find the slope of the line $y = mx$ that divides R into two equal areas.



10. Find the value of the integral: $\displaystyle\int_{\frac{p}{6}}^{\frac{p}{3}} \frac{\tan^2 q}{\tan^2 q + \cot^2 q}\, dq$.