LVAIC Mathematics Contest - Oct. 19, 2002

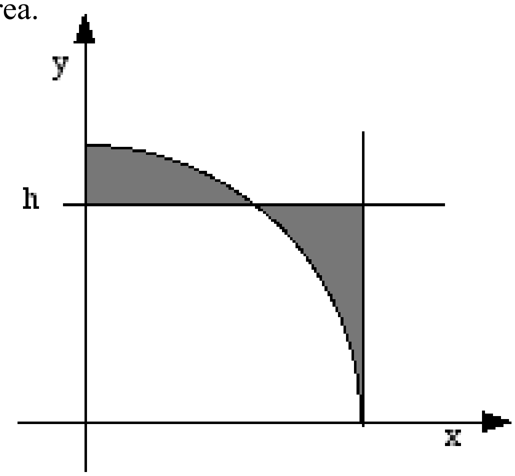
Do as many of these problems as you can. No calculators or notes are allowed. Your solutions must be complete and your work justified to receive full credit. Write up each solution on a separate sheet of paper!

1. Here is a sequence of numbers: 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 1, …

   What is the 2002nd number on this list?

2. A math teacher wrote a large integer on the board, and her students commented on the number: the 1st student said that the number was divisible by 2, the 2nd student said it was divisible by 3, and so on, until the 30th student said it was divisible by 31. The proud teacher told the class 'Good work, but two of the statements were incorrect, and the incorrect statements were made consecutively.' Find the two erroneous statements.

3. In the figure, one quarter of the circle $x^2 + y^2 = 1$ is drawn. Find the height h which minimizes the shaded area.



4. Suppose a and b are positive real numbers. Show that $a(1 - b)$ and $b(1 - a)$ cannot both be greater than 1/4.

5. Suppose $p > 3$ is a prime number. Show that $p^2 - 1$ is always a multiple of 12.

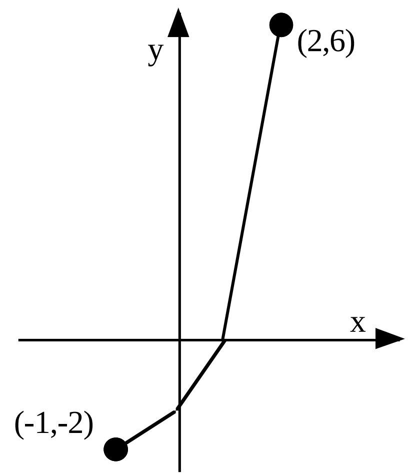
6. You are given 2002 numbers: 1001 of them are 1's and 1001 are –1's. All of these 1's and –1's are placed around a circle; write $a_1, a_2, \ldots, a_{2002}$ for the numbers as they are placed in order around the circle. Let $S = a_1a_2a_3 + a_2a_3a_4 + \ldots + a_{2002}a_1a_2$.

   a) Show that S must be an even number.

   b) Determine the maximum and minimum possible values for S, with proof.

More goodies on the back!

7. A mysterious substance, sloperium, coats the x and y axes. This substance has the following effect: when a particle moving in a line strikes either axis, the slope of the particle's path is increased by 1. Starting from the point (-1,-2), what point on the negative y-axis should you aim for in order to hit a target at the point (2,6)?



8. Consider three orderings of the integers from 1 to 100:

$$a_1, a_2, \ldots, a_{100} \qquad b_1, b_2, \ldots, b_{100} \qquad c_1, c_2, \ldots, c_{100}$$

Let s = the minimum value of

$$\{\ |a_1 - b_1|,\ |a_1 - c_1|,\ |b_1 - c_1|,\ |a_2 - b_2|,\ |a_2 - c_2|,\ |b_2 - c_2|, \ldots, \\ |a_{100} - b_{100}|,\ |a_{100} - c_{100}|,\ |b_{100} - c_{100}|\ \}.$$

Show that $s \leq 33$, no matter what the orderings were. Can $s = 33$?

9. N students and N professors are sitting in a circle around a large table. Some of the students and professors are liars (they *always* lie) and some *always* tell the truth. Suppose the number of lying students is the same as the number of lying professors. When asked 'What is your right-hand neighbor at the table?' each person responds "Student." Show that N must be even.

10. The lattice points in the plane are just the points in which both coordinates are integers. Suppose circles of radius r are drawn using every lattice point as a center. What is the smallest value of r that ensures that any line of slope 2/3 will intersect some of the circles?