LVAIC Mathematics Contest - Oct. 13, 2001

Do as many of these problems as you can. No calculators or notes are allowed. Your solutions must be complete and your work justified to receive full credit. Write up each solution on a separate sheet of paper!

1. How many positive integers exactly divide the number 2001^{2001}?

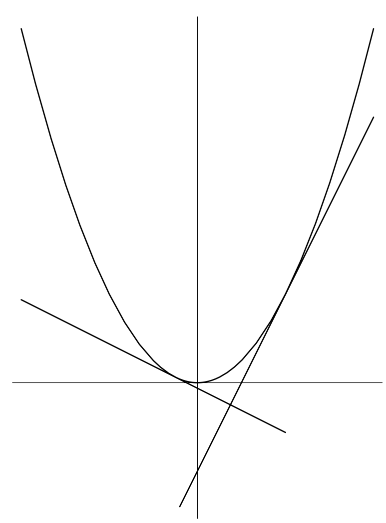
2. For the given table, you can start at any letter, then move horizontally or vertically one letter at a time (up or down, left or right). How many different paths in the table will spell the word LVAIC (the letters must occur in the correct order)? Explain!

L	**V**	**A**	**I**	**C**
V	**L**	**V**	**A**	**I**
A	**V**	**L**	**V**	**A**
I	**A**	**V**	**L**	**V**
C	**I**	**A**	**V**	**L**

3. A two-person game is played as follows: 2n numbered balls are placed in a basket. Player 1 and 2 alternate selecting balls, each player keeping all the balls they select in a pile. You lose the game when you select ball number k if k + r = 2n+1, where r is the number on one of the balls in your opponent's pile. If all the balls are exhausted before this happens, it's a draw. Depending on n and assuming each player follows an optimum strategy, who will win? What is the strategy?

4. The numbers from 1 to 2001 are listed in the following weird order: First, list all the odd numbers in increasing order, then list all numbers of the form 2*k*, where *k* is an odd number, then all numbers of the form 4*k*, where *k* is odd, then all numbers of the form 8*k*, where *k* is odd, and so on. Finally, list the remaining numbers (those that do not fall into any of the previous categories) in increasing order. What is the last number on the list?

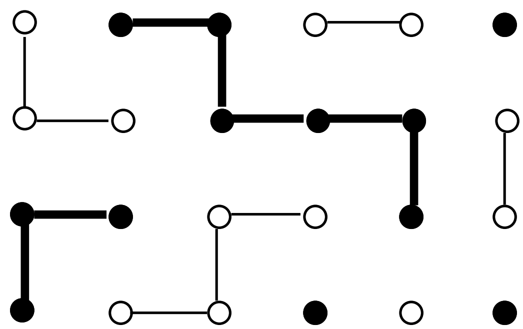
5. Call a point *good* if the two tangent lines from the point to the parabola $y = x^2$ are perpendicular. Find a good description of all good points in the *xy*-plane. (See the picture for a picture to look at.)



(Turn over for more problem solving pleasure.)

6. For each point (m, n) in the xy-plane (where m and n are integers), draw a circle of radius $\frac{1}{2^{|m|+1}3^{|n|+1}}$. What is the total area enclosed by all these circles?

7. I've drawn 24 points in a 6 x 4 array. Each point is either hollow or solid. Further, each row and each column of the array have the same number of hollows and solids. I've drawn line segments (either horizontal or vertical, but not diagonal) between two neighboring points if they are of the same type, and call the resulting segments hollow or solid. Show that no matter how the hollow / solid points are distributed in the array (keeping the requirement that each row and each column have the same number of hollows and solids), the number of hollow segments equals the number of solid segments.



8. Find the limit: $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2}$

9. Suppose $a_1, \ldots, a_n$ is a sequence of numbers, each of which is 1 or –1. What is the minimum possible value of the sum $\sum_{i<j} a_i a_j$ $(= a_1a_2 + a_1a_3 + \ldots + a_{n-1}a_n)$?

10. Let P be the point (1,1), and consider the 2 concentric circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Draw a ray from P so that it intersects each circle at the points Q and R, as shown. Find the ray that maximizes the length of the segment QR. For that ray, give the coordinates of the points Q and R.