LVAIC Mathematics Contest - Nov. 13, 1999

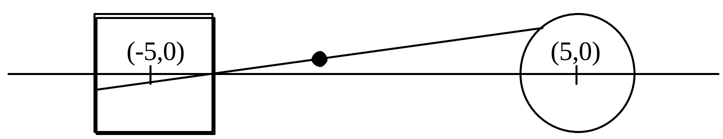
Do as many of these problems as you can. No calculators or notes are allowed. Your solutions must be complete and your work justified to receive full credit. Write up each solution on a separate sheet of paper!

1. Which number is larger, 19^{99} or 99^{19}? Explain carefully.

2. You've graduated from college and are now making megabucks designing new games. One of your workers has the following idea for a geometric puzzle: You are given thirteen 2 in. x 1 in. x 1 in. blocks. Your goal is to assemble them into a 3 in. x 3 in. x 3 in. cube, except the 1 in. x 1 in. x 1 in. block at the center of the cube will be missing. Is it possible to solve such a puzzle? If so, explain carefully how to solve it and how you will give the worker a promotion. If not, explain why it's not possible and how your worker should be fired.

3. Here is a sequence of positive integers 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, …What is the 1999^{th} number in this list?

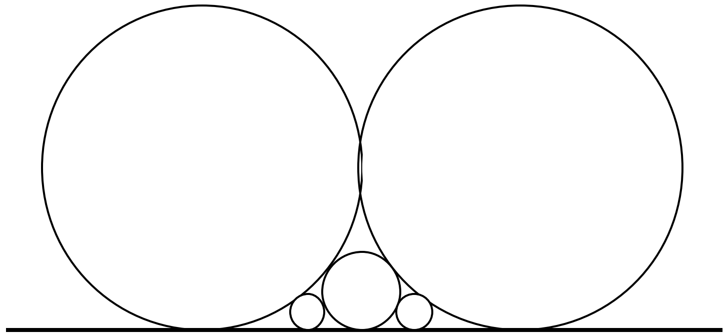
4. Squaring the circle? Circling the square? A circle of radius 2 is centered at the point (5,0), and a 2 x 2 square is placed in the plane so its corners are at (-4, 1), (-4, -1), (-6, 1) and (-6, -1). A line segment is drawn joining one point on the circle and one point on the square, and the midpoint is marked, as in the picture. This is done for every possible choice of two points, one on the circle and the other on the square, and each time the midpoint is marked. What is the resulting figure? Give a careful drawing, showing all boundaries, and compute the area of the figure.



5. Liz is 10 blocks west of a river that runs north-south in a city with perfectly square blocks. She wants to walk to the river (to sit quietly and work math problems), and has decided to do that by walking either north or east at each corner, however, she will never walk north two blocks in a row. She plans on stopping as soon as she reaches the river. How many ways can she walk to the river?

6. Let $a_1, a_2, \ldots, a_{1999}$ be some rearrangement of the numbers 1, 2, …, 1999. Let P = $(a_1 - 1)(a_2 - 2)(a_3 - 3) \ldots (a_n - n)$. Choose one (and only one) of the following
   a. P must be an even integer.
   b. P must be an odd integer
   c. Sometimes P is even and sometimes it's odd.

7. In the picture, the largest circles have radius equal to 1. All circles are tangent to each other and the line. Find the radius of the smallest circles.

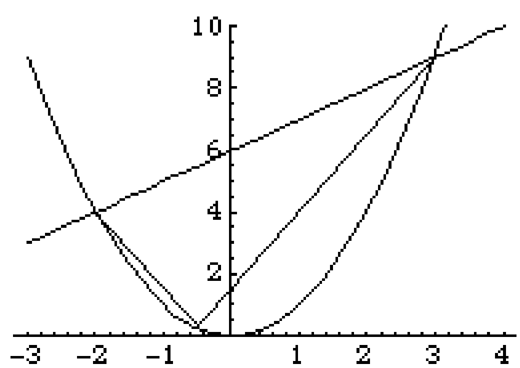


8. Rebecca and Hannah are having a discussion about math.

Hannah: Here's a neat number thing. Take all of the positive integers and put them into 3 piles, and no pile can be empty.
Rebecca: So every positive integer is in exactly one of the piles?
Hannah: Yes. Let's call the three piles A, B, and C. Here's the tricky part. You have to do it so that if you add any number in A to any number in B, you get some number in C.
Rebecca: That's not hard.
Hannah: I'm not finished; if you add any number in B to any number in C, you get some number in A.
Rebecca: Let me guess: you also want it so that if you add any number in A to any number in C, you get some number in B.
Hannah: Yes, that's what you want.
Rebecca: That's impossible.
Hannah: I think it can be done.

Who's right? If Hannah is right, give an example of sets A, B, and C (with every positive integer in exactly one of these non-empty sets) satisfying the requirements. If Rebecca is right, show why it's impossible to make 3 such sets.

9. The line $y = x+ 6$ meets the parabola $y = x^2$ in two points, as in the picture. Form a triangle using these two points and a third point on the parabola below this line. Find the third point which maximizes the area of the triangle.



10. Choose a positive integer n less than 1000 and compute the palindrome of n (formed by reversing the digits). For example, if n = 482, then the palindrome of n is 284. Now compute the absolute value of the difference between n and

its palindrome. Repeat this process over and over. Prove that you will eventually get 0, no matter what integer n you started with.