LVAIC Mathematics Contest - Oct. 24, 1998

Do as many of these problems as you can. No calculators or notes are allowed. Your solutions must be complete and your work justified to receive full credit. Write up each solution on a separate sheet of paper!

1. How many positive integer divisors of 10^{1998} are there?

2. Here is a sequence of numbers:

   $a_1 = 1$ $\quad a_2 = 2 + 3$ $\quad a_3 = 4 + 5 + 6$ $\quad a_4 = 7 + 8 + 9 + 10$
   …

   Find a formula for a_n.

3. Find the highest power of 3 which divides 1998!

4. A cube of side length 4 has 9 spheres inside it; 8 spheres of radius 1 are placed in the 8 corners and 1 sphere is in the center. The radius of the sphere in the center is chosen so that the center sphere is tangent to each of the other 8 spheres. Find the radius of the center sphere.

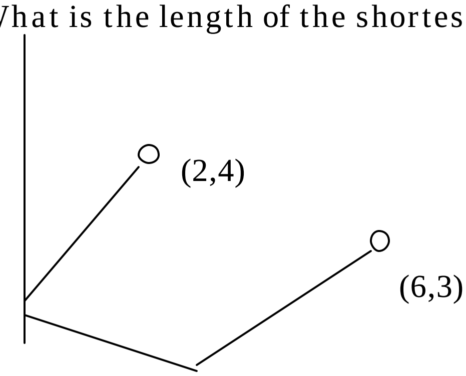
5. Let $f(x) = x^2 + bx + c$, where b and c are real numbers between –1 and 1. What is the probability that both roots of $f(x)$ are positive real numbers?

6. Here is a conversation between Hannah and Rebecca:

Hannah: I just learned in school that any 3 non-collinear points lie on a circle.
Rebecca: Big deal – everyone knows that. I'll bet you didn't know it also works for squares.
Hannah: I don't believe it. I'll bet I can find 3 points which can never be on the sides of any square.
Rebecca: You're wrong – any three points are always on the sides of some square.

   Who's right? Given any 3 points, is there always a square which has those 3 points somewhere on its sides? If so, show how to construct a square no matter where the points are. If not, give an example (with proof) of 3 points which do not lie on the sides of any square.

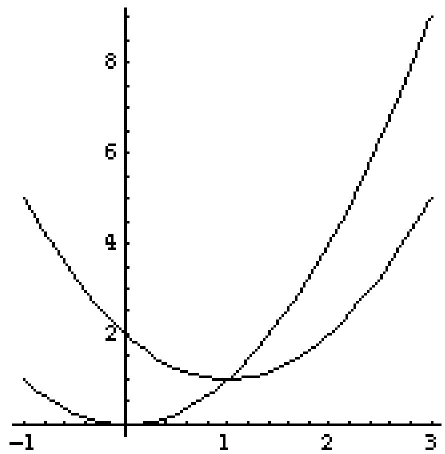
7. We need to connect the points (6,3) and (2,4) by line segments which meet the x-axis and y-axis, as in the picture. What is the length of the shortest such path joining these two points?



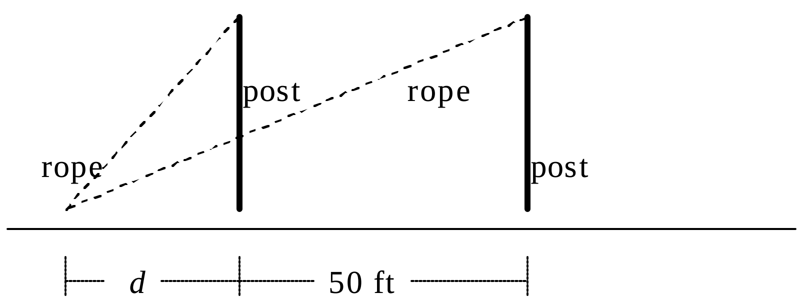
(See other side for the rest of the exam.)

8. Here's a game. Given any positive integer n, first add its digits, then double this sum. Now, apply this process to the result over and over again. For example, if n = 156, then the process first gives 24, then gives 12, and so on. Determine all possible long-term behaviors for a given initial n. Your solution should include a complete description of what the ultimate outcome of this process will be for any initial n, with justification, of course.

9. The parabolas $y = x^2$ and $y = x^2 - 2x + 2$ are graphed to the right. Find the equation of a line which is tangent to both curves simultaneously.



10. Two vertical posts are 25 feet tall and 50 feet apart. A 100 foot rope joins the top of one post with the top of the other. The rope is pulled taut, as in the picture, with the rope touching the ground at one point. How far is the part of the rope touching the ground from the closest pole? (This is the distance labeled d in the picture.)



11. Let $a_0 = 1/2$ and let $a_{n+1} = 1 - a_n^2$. Find $\lim_{n\to\infty} a_{2n+1}$.

12. Find all polynomials $p(x)$ so that $p(0) = 0$ and $p(x^2+1) = \frac{1}{2}p(x)^2 + 2$.