RESEARCH STATEMENT

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I am working in the area of functional analysis and operator theory, on problems related to the geometry of Banach spaces. To be more concrete, my current interest is in norm-attaining and numerical radius attaining operators, especially, in relation to the **Bishop-Phelps-Bollobás theorem**. This theorem is the extension of the celebrated Bishop-Phelps theorem, a powerful tool that was the origin of a series of "perturbed optimization" results for different classes of functions [1]. Specifically, it states that the set of bounded linear functionals that attain their maximum on a closed bounded convex set in a Banach space is dense in the dual. A natural question asked at the end of the publication by Bishop and Phelps in 1961 addressed whether this result would also be true for operators on Banach spaces [5]. It is not surprising that in general, the answer is "no". For some Banach spaces there is an operator that almost attains its norm but it cannot be approximated by a norm-attaining operator [15]. But by imposing certain conditions on either the domain space or on the range, it can be assured that this never happens. Investigating these conditions has generated a great deal of interest in many analysts over the course of the following forty years.

The main part of my work is devoted to researching this question from the point of view of the Bishop-Phelps-Bollobás theorem, which deals with simultaneously approximating both functionals and the points at which they almost attain their norms by norm-attaining functionals and the points at which they attain their norm. It is equivalent to two well-known variational principles: Ekeland's variational principle and the Brønsted-Rockafellar principle. But in essence, this could be thought of as an "**infinite dimensional optimization problem**". Here is an intuitive explanation. Suppose we have a continuous (real)-valued function f on a closed bounded set $C \subset X$. If X is finite-dimensional, then C would be compact, and the maximum (or minimum) point of f would exist. In the infinite dimensional case, the maximum (minimium) points do not necessarily exist. Then the Bishop-Phelps-Bollobás theorem or, the Brønsted-Rockafellar principle, provide a solution: if f almost attains its maximum at some $x \in X$, then there exists a bounded linear functional x^*, of a norm as small as needed, so that the perturbation $f + x^*$ attains the maximum at some $x_0 \in X$, and x is very close to x_0 [17].

Several directions of my current and future work include:

- The Bishop-Phelps-Bollobás theorem in the setting of operators on Banach spaces;
- Norm-attaining operators and functionals on complex Banach spaces;
- Constructive version of the Bishop-Phelps-Bollobás theorem, and as the consequence, a constructive version of the Bishop-Phelps theorem;
- Numerical radius attaining operators.

The details are presented below.

The Bishop-Phelps-Bollobás property for operators. While the denseness of norm-attaining operators has been studied extensively and is a well-known topic, this approach is relatively new. In 2008, M. Acosta et al. introduced the above theorem to the operator setting. A pair of Banach spaces is said to have the *Bishop-Phelps-Bollobás property for operators* if given $T : X \to Y$, a bounded linear operator, $\|T(x_0)\| \approxeq \|T\|$ for some $\|x_0\| = 1$, then there is a norm-attaining operator $S : X \to Y$ and a point $u_0 \in X$, $\|u_0\| = 1$ with $\|S(u_0)\| = \|S\| = \|T\|$ such that $u_0 \approxeq x_0$ and $S \approxeq T$ [2].

There are a number of reasons why I am particularly interested in this area. First of all, the Bishop-Phelps-Bollobás theorem is stronger than the Bishop-Phelps theorem. Hence, whenever a pair of spaces (X, Y) has a set of norm-attaining operators which is not dense in the set of bounded linear operators, it automatically fails the Bishop-Phelps-Bollobás property. Several classes of examples from [2] show, for instance, (l_1, Y) possesses this property when Y is a uniformly convex space; a finite-dimensional space;

$L_1(\mu)$, where μ is a σ-finite measure; or a space of continuous functions $C(K)$ for a compact Hausdorff space K. Another positive example noted was (l_n^∞, Y) for Y uniformly convex. However, the tools used to prove it rely on n and would not work, for example, for the space of sequences converging to zero, c_0. In joint work with R. Aron (Kent State University) and B. Cascales (University of Murcia, Spain), we proved that a pair (c_0, Y) has the Bishop-Phelps-Bollobás property for every Banach space Y [3].

In my dissertation, I look at the case of bounded linear operators on the space of continuous functions. For full generality, let L be a locally compact Hausdorff space. In [3], we considered the case of bounded linear operators $T : X \to C_0(L)$, and showed that whenever T is an Asplund operator the desired approximation is possible. An operator $T \in L(X, Y)$ is called an Asplund operator if it can be factored through an Asplund space, *i.e.*, there are an Asplund space Z and operators $T_1 \in L(X, Z), T_2 \in L(Z, Y)$ such that $T = T_2 \circ T_1$.

Theorem 1.1. [3] *Let $T : X \to C_0(L)$ be an **Asplund operator** with $\|T\| = 1$. Suppose that $0 < \varepsilon < \frac{1}{2}$ and $x_0 \in S_X$ are such that*

$$\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}.$$

*Then there are $u_0 \in S_X$ and an **Asplund operator** $S : X \to C_0(L)$, satisfying*

$$\|S(u_0)\| = 1 = \|S\|, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| \leq 3\varepsilon.$$

I believe this theorem is useful in a variety of ways. A famous result by W. Davis, T. Figiel, W. Johnson, and A. Pełczyński [11] states that every weakly compact operator can be factored through a reflexive space which is, in fact, Asplund. Hence, the theorem above holds for weakly compact operators, compact operators, finite-rank operators, and p-summing operators. Moreover, S is constructed in such a way that it is in the same operator sub-ideal as T. Also, if either X is an Asplund space or L is a scattered locally compact Hausdorff space, then the condition of T being Asplund can be dropped and $(X, C_0(L))$ has the Bishop-Phelps-Bollobás property for operators. For such pairs, our result strengthens the result by J. Johnson and J. Wolfe that the set of norm-attaining operators $T : X \to C(K)$, where K is a compact Hausdorff space, is dense [13].

Recently Theorem 1.1 has been taken one step further and proved for Asplund operators $T : X \to \mathfrak{U}$, where $\mathfrak{U} \subset C(K)$ is a uniform algebra [3]. My short-term goal is to investigate this property for operators defined on the space of continuous functions, i.e. $T : C(K) \to X$ or $T : C(K) \to C(S)$.

Norm-attaining operators and complex Banach spaces. If C is a subset of a Banach space, a point x is called a support point of C if there is a norm-attaining functional f which attains its supremum on C at x. Then f is called a support functional. V. Klee in [14] asked if each closed bounded convex subset of a Banach space has a support point. The answer was given in the Bishop-Phelps theorem, that in fact, the set of support functionals is norm dense.

Even though the original Bishop-Phelps theorem was proved for a closed bounded convex set C in a *real* Banach space, it is still true if C is the unit ball of a *complex* Banach space. J. Bourgain showed that if X is a complex Banach space with the Radon-Nikodym property [6], then the unit ball could be replaced by an arbitrary closed convex bounded set. Might this condition be removed and this statement be true in general? This question remained open for almost twenty years, until V. Lomonosov [16] constructed a very "pathological" counterexample: a complex Banach space and a closed convex bounded set C with no support points whatsoever. Thus the general Bishop-Phelps theorem cannot be extended to complex spaces.

As a part of my ongoing and future research, I study what goes so fundamentally "wrong" for the norm-attaining operators in complex Banach spaces allowing such an example to be produced. We have been working with A. Guirao (Polytechnic University of Valencia, Spain) on a *complex*-valued Bishop-Phelps-Bollobás theorem, and in [12], we obtained a result for the specific case of the unit ball of ℓ_1.

In this paper, another important issue is addressed. The original proof of the Bishop-Phelps theorem relies on the Hahn-Banach theorem and, therefore, solely speaks of the existence of an approximating norm-attaining functional. The theorem for ℓ_1 mentioned above is a **constructive** version, and it provides a concrete approximating functional and the point. A more general approach in this direction is not yet known. Since the Bishop-Phelps theorem is used extensively, I am planning to work on a constructive approach in order to get a general "recipe" of the approximating pair of the norm-attaining functional and the point.

Numerical radius attaining operators. There is a strong parallel between the denseness of norm-attaining and numerical radius attaining operators. The numerical range of a bounded linear operator $T : X \to X$ is a set of scalars

$$V(T) = \{x^*(Tx)\colon\ x \in X, x^* \in X^*, \|x\| = \|x^*\| = x^*(x) = 1\}.$$

The numerical radius is defined as $\nu(T) = \sup\{|\lambda|\colon\ \lambda \in V(T)\}$. When the supremum is actually a maximum, we say that an operator T **attains its numerical radius** $\nu(\mathbf{T})$. It is easy to produce even a self-adjoint operator which does not attain its numerical radius. B. Sims initiated the study of denseness of the numerical radius attaining operators[19]. Some examples of spaces whose set of numerical radius attaining operators is indeed dense include c_0, ℓ_1, $C(K)$, $L_1(\mu)$, and uniformly smooth spaces [8, 9, 7].

Yet these two problems, the denseness of norm-attaining and numerical radius attaining operators, are independent. The numerical radius $\nu(T)$ is a seminorm on the set of bounded linear operators $L(X)$, and for every $T \in L(X)$, $\nu(T) \leq \|T\|$. Suppose that for some operator $T \in L(X)$, $\nu(T) = \|T\|$. Then if T attains $\nu(T)$, then it attains $\|T\|$ as well. On the other hand, if T attains the norm, it may not attain its numerical radius. Now supposing that $\nu(T) < \|T\|$, then T may attain its numerical radius, yet fail to attain the norm [18].

Because of the interconnection between the norm-attaining and numerical radius attaining operators, it makes sense to consider the approximation by numerical radius attaining operators with consideration of the point at which the numerical radius is attained. In joint research with A. J. Guirao, we introduce a new property called the **"Bishop-Phelps-Bollobás property for numerical radius"** and provide the first few examples of spaces satisfying it, such as ℓ_1 and c_0 [12]. Another example appeared recently in [4]: $C(K)$ has the above property whenever K is metrizable.

As a newly introduced concept, it has a lot of opportunities for research. For instance, from the start we have an example of a space lacking this property: $X = c_0 \bigoplus Y$, where Y is a strict renorming of c_0. This example was used by R. Payá in [18] as a space whose set of numerical radius attaining operators is not dense. Also, it was previously used by J. Lindenstrauss in [15] to show that the set of bounded linear operators $T : X \to X$ which attain their norms is not dense. I am very interested in getting an example of a space that has a dense set of numerical radius attaining operators, yet failing the Bishop-Phelps-Bollobás property for numerical radius.

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