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The Bishop–Phelps–Bollobás property for numerical radius in $\ell_1(\mathbb{C})$

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Abstract. We show that the set of bounded linear operators from X to X admits a Bishop–Phelps–Bollobás type theorem for numerical radius whenever X is $\ell_1(\mathbb{C})$ or $c_0(\mathbb{C})$. As an essential tool we provide two constructive versions of the classical Bishop–Phelps–Bollobás theorem for $\ell_1(\mathbb{C})$.

1. Introduction. The Bishop–Phelps theorem states that the norm attaining functionals on a Banach space X are dense in its dual space X^*. In 1970, B. Bollobás extended this result in a quantitative way in order to work on problems related to the numerical range of an operator [Bol70]. One of the versions of his extension is presented below:

THEOREM 1.1. *Let X be a Banach space. Given $\varepsilon > 0$, if $x \in X$ and $x^* \in X^*$ satisfy $\|x\| = \|x^*\| = 1$ and $x^*(x) \geq 1 - \varepsilon^2/2$, then there exist elements $x_0 \in X$ and $x_0^* \in X^*$ such that $\|x_0\| = \|x_0^*\| = x_0^*(x_0) = 1$,*

$$\|x - x_0\| \leq \varepsilon \quad \textit{and} \quad \|x^* - x_0^*\| \leq \varepsilon.$$

However, the known proofs of this fact have an existence nature—they are based on the Hahn–Banach extension theorem, the Ekeland variational principle or Brøndsted–Rockafellar principle. In this paper we construct, as a necessary tool for our main results, explicit expressions for the approximating pair (x_0, x_0^*) when $X = \ell_1(\mathbb{C})$ (see Theorems 2.4 and 2.6).

Paralleling the research of norm attaining operators initiated by Lindenstrauss in [Lin63], B. Sims raised the question of the norm denseness of the set of numerical radius attaining operators (see [Sim72]). Partial positive results have been proved. Due to their importance we emphasize the results of M. Acosta in her Ph.D. thesis [Aco90], where a systematic study of the problem was initiated, the renorming result in [Aco93], and joint findings of this author with R. Payá [AP89, AP93]. Prior to them, I. Berg and

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B. Sims [BS84] gave a positive answer for uniformly convex spaces, and C. S. Cardassi obtained positive answers for ℓ_1, c_0, $C(K)$, $L_1(\mu)$, and uniformly smooth spaces [Car85a, Car85b, Car85c].

Using a renorming of c_0, R. Payá provided an example of a Banach space X such that the set of numerical radius attaining operators on X is not norm dense, answering in the negative Sims' question (see [Pay92]). In the same year, M. Acosta, F. Aguirre, and R. Payá [AAP92] gave another counterexample: $X = \ell_2 \oplus_\infty G$, where G is the Gowers space.

Recently, M. Acosta et al. [AAGM08] studied a new property, called the *Bishop–Phelps–Bollobás property for operators*, BPBp for short. A pair of Banach spaces (X, Y) has the BPBp if a "Bishop–Phelps–Bollobás" type theorem can be proved for the set of operators from X to Y. This property implies, in particular, that the norm attaining operators from X to Y are dense in the whole space of continuous linear operators $\mathfrak{L}(X, Y)$. However, as shown in [AAGM08], the converse is not true. Consequently, the BPB property is more than a quantitative tool for studying the density of norm attaining operators.

We investigate here an analogue of the Bishop–Phelps–Bollobás property for operators but in relation to numerical radius attaining operators. We call it the *Bishop–Phelps–Bollobás property for numerical radius*, BPBp-ν for short. The relation between norm attaining and numerical radius attaining operators is far from being clear, although the existence of an interconnection is evident. Accordingly, our goals in this paper are to define this new property (see Definition 1.2 below) and to show that $\ell_1(\mathbb{C})$ and $c_0(\mathbb{C})$ satisfy it (see Theorems 3.1 and 4.1). This brings an extension as well as a quantitative version of C. S. Cardassi's results in [Car85b].

Observe that the counterexamples provided in [AAP92] and [Pay92] imply, in particular, that there exist Banach spaces failing the Bishop–Phelps–Bollobás property for numerical radius.

Given a Banach space $(X, \|\cdot\|)$, we denote as usual by S_X and B_X, respectively, the unit sphere and the unit ball of X. By X^* we represent its dual, endowed with its standard norm $\|x^*\| = \sup_{x \in B_X}\{|x^*(x)|\}$, and we set

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} \colon x^*(x) = 1\}.$$

Given $x \in S_X$ and $x^* \in S_{X^*}$, we set

$$\pi_1(x^*) = \{x \in S_X \colon x^*(x) = 1\}.$$

By $\mathfrak{L}(X)$ we denote the Banach space of all linear and continuous operators from X into X endowed with its natural norm $\|T\| = \sup_{x \in B_X}\{\|Tx\|\}$. For a given $T \in \mathfrak{L}(X)$, its *numerical radius* $\nu(T)$ is defined by

$$\nu(T) = \sup\{|x^*(Tx)| \colon (x, x^*) \in \Pi(X)\}.$$

It is well known that the numerical radius of a Banach space X is a continuous seminorm on X which is, in fact, an equivalent norm when X is complex. In general, there exists a constant $n(X)$, called the *numerical index* of X, such that

$$n(X)\|T\| \le \nu(T) \le \|T\| \quad \text{for all } T \in \mathfrak{L}(X).$$

Our interest in this paper is in spaces of numerical index 1, $n(X) = 1$, where the norm and the numerical radius coincide. For background in numerical radius we refer the reader to the monographs [BD71, BD73], and for numerical index to the survey [KMP06].

We say that $T \in \mathfrak{L}(X)$ *attains its numerical radius* if there exists $(x, x^*) \in \Pi(X)$ such that $|x^*(Tx)| = \nu(T)$. The set of numerical radius attaining operators will be denoted by $\mathrm{NRA}(X) \subset \mathfrak{L}(X)$.

Definition 1.2 (BPBp-ν). A Banach space X is said to have the *Bishop–Phelps–Bollobás property for numerical radius* if for every $0 < \varepsilon < 1$, there exists $\delta > 0$ such that for any given $T \in \mathfrak{L}(X)$ with $\nu(T) = 1$ and a pair $(x, x^*) \in \Pi(X)$ satisfying $|x^*(Tx)| \ge 1 - \delta$, there exist $S \in \mathfrak{L}(X)$ with $\nu(S) = 1$ and a pair $(y, y^*) \in \Pi(X)$ such that

$$\nu(T - S) \le \varepsilon, \quad \|x - y\| \le \varepsilon, \quad \|x^* - y^*\| \le \varepsilon \quad \text{and} \quad |y^*(Sy)| = 1. \tag{1.1}$$

Observe that if X is a Banach space with $n(X) = 1$, then the seminorm $\nu(\cdot)$ can be replaced by $\|\cdot\|$ in the definition above. Note that all the spaces studied in this paper have numerical index 1.

Notation and terminology. Throughout this paper $\arg(\cdot)$ stands for the function which sends a non-zero complex number z to the unique $\arg(z) \in [0, 2\pi)$ such that $z = |z|e^{\arg(z)i}$. For convenience we extend this function to $\mathbb{C}$ by writing $\arg(0) = 0$. Following the standard notation, let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ be, respectively, the real and imaginary part of the complex number $z \in \mathbb{C}$.

All along Sections 2 to 4, the spaces ℓ_1, ℓ_∞, and c_0 stand respectively for $\ell_1(\mathbb{C})$, $\ell_\infty(\mathbb{C})$, and $c_0(\mathbb{C})$. The standard basis of ℓ_1 is denoted by $\{e_n\}_{n\in\mathbb{N}}$, and its biorthogonal functionals by $\{e_n^*\}_{n\in\mathbb{N}}$. Given a sequence $\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ and a complex function $f\colon \mathbb{C} \to \mathbb{C}$ we write $f(\xi)$ for the sequence $(f(\xi_j))_{j\in\mathbb{N}}$.

The following sets will be of help in the formulation of the results and proofs. Given $x = (x_j)_{j\in\mathbb{N}} \in \ell_1$, $\varphi = (\varphi_j)_{j\in\mathbb{N}} \in \ell_\infty$ we define

$$\mathcal{N}_{(x,\varphi)} = \{j \in \mathbb{N}\colon \varphi_j x_j = |x_j|\}, \tag{1.2}$$
$$\mathrm{supp}(x) = \{j \in \mathbb{N}\colon |x_j| \ne 0\}.$$

For $r > 0$ we consider

$$\mathcal{A}_\varphi(r) = \{j \in \mathbb{N}\colon |\varphi_j| \ge 1 - r\}, \tag{1.3}$$

$$\mathcal{P}_{(x,\varphi)}(r) = \{j \in \mathrm{supp}(x)\colon \mathrm{Re}(\varphi_j x_j) \ge (1 - r)|x_j|\}. \tag{1.4}$$

Observe that $\mathcal{P}_{(x,\varphi)}(r) \subset \mathcal{A}_\varphi(r)$ and that if $x_j \geq 0$ for all $j \in \mathbb{N}$ (we then say that x is *positive*) then

$$\mathcal{P}_{(x,\varphi)}(r) = \{j \in \operatorname{supp}(x)\colon\ \operatorname{Re}(\varphi_j) \geq (1-r)\}.$$

For a given set Γ, a subset $A \subset \Gamma$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we denote by $\mathbb{1}_A$ the characteristic function of A, that is, the element in $\mathbb{K}^\Gamma$ such that $(\mathbb{1}_A)_\gamma = 1$ if $\gamma \in A$ and $(\mathbb{1}_A)_\gamma = 0$ otherwise.

2. The Bishop–Phelps–Bollobás theorem in $\ell_1(\mathbb{C})$. In this section we present two constructive versions of Theorem 1.1, which are the main tools in the proofs of Theorems 3.1 and 5.1.

LEMMA 2.1. *Let $(x,\varphi) \in S_{\ell_1} \times S_{\ell_\infty}$. Then $x \in \pi_1(\varphi)$ if and only if $\mathcal{N}_{(x,\varphi)} = \mathbb{N}$.*

Proof. Given a pair $(x,\varphi) \in S_{\ell_1} \times S_{\ell_\infty}$ satisfying $\mathcal{N}_{(x,\varphi)} = \mathbb{N}$, one can compute $\varphi(x) = \sum_{j\in\mathbb{N}} \varphi_j\, x_j \overset{(1.2)}{=} \sum_{j\in\mathbb{N}} |x_j| = \|x\| = 1$, which implies that $(x,\varphi) \in \Pi(\ell_1)$.

Conversely, assume that $(x,\varphi) \in \Pi(\ell_1)$. Then

$$1 = \operatorname{Re}(\varphi(x)) = \sum_{j\in\mathbb{N}} \operatorname{Re}(\varphi_j x_j) \leq \sum_{j\in\mathbb{N}} |\varphi_j x_j| \leq \sum_{j\in\mathbb{N}} |x_j| = 1,$$

which implies that $\operatorname{Re}(\varphi_j x_j) = |\varphi_j x_j| = |x_j|$ for $j \in \mathbb{N}$. Thus, $\varphi_j x_j = |x_j|$ for every $j \in \mathbb{N}$, which finishes the proof. ■

Lemma 2.1 is essential to the proofs of Theorems 2.4 and 2.6. A glance at it gives the following easy result regarding the norm attaining functionals on ℓ_1, $\mathrm{NA}(\ell_1)$.

COROLLARY 2.2. $\mathrm{NA}(\ell_1) = \{\varphi \in \ell_\infty\colon \exists n \in \mathbb{N}$ *with* $|\varphi_n| = \|\varphi\|\}$.

The following lemma is an adaptation of [AAGM08, Lemma 3.3] to our notation.

LEMMA 2.3. *Let $(x,\varphi) \in B_{\ell_1} \times B_{\ell_\infty}$ and $0 < \delta < 1$ be such that $\operatorname{Re}(\varphi(x)) \geq 1-\delta$. Then for every $\delta < r < 1$ we have*

$$\|\operatorname{Re}(e^{\arg(\varphi)i}x) \cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\| \geq 1 - \delta/r.$$

Proof. By assumption, we have

$$\begin{aligned}
1-\delta \leq \operatorname{Re}(\varphi(x)) &= \sum_{j\in\mathbb{N}} \operatorname{Re}(\varphi_j x_j) = \sum_{j\in\mathbb{N}} |\varphi_j| \operatorname{Re}(e^{\arg(\varphi_j)i} x_j) \\
&\leq \sum_{\mathcal{P}_{(x,\varphi)}(r)} \operatorname{Re}(e^{\arg(\varphi_j)i} x_j) + (1-r) \sum_{\mathbb{N}\setminus\mathcal{P}_{(x,\varphi)}(r)} |x_j| \\
&\leq r \sum_{\mathcal{P}_{(x,\varphi)}(r)} |\operatorname{Re}(e^{\arg(\varphi_j)i} x_j)| + (1-r),
\end{aligned}$$

which implies that

$$\begin{aligned}\|\mathrm{Re}(e^{\arg(\varphi)i}x)\,\mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\| &= \sum_{j\in\mathcal{P}_{(x,\varphi)}(r)} |\mathrm{Re}(e^{\arg(\varphi_j)i}x_j)| \\ &\geq 1-\delta/r. \ \blacksquare\end{aligned}$$

Observe that the previous lemma implies, in particular, that

$$\|x\cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(r)}\| \geq 1-\delta/r.$$

We next present the two constructive versions of the Bishop–Phelps–Bollobás theorem.

2.1. First constructive version

THEOREM 2.4. *Given $(x,\varphi)\in B_{\ell_1}\times B_{\ell_\infty}$ and $0<\varepsilon<1$ such that $\mathrm{Re}(\varphi(x))\geq 1-\varepsilon^3/4$, there exists $(x_0,\varphi_0)\in \Pi(\ell_1)$ such that $\|x-x_0\|\leq\varepsilon$ and $\|\varphi-\varphi_0\|\leq\varepsilon$. Moreover, we can take*

$$x_0 = \|x\cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)}\|^{-1}\cdot x\cdot \mathbb{1}_{\mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)}. \tag{2.1}$$

Proof. Set $P=\mathcal{P}_{(x,\varphi)}(\varepsilon^2/2)$ (see definition (1.4)). Applying Lemma 2.3 with $\delta=\varepsilon^2/2$ and $r=\varepsilon$ gives

$$M := \|x\cdot \mathbb{1}_P\| \geq 1-\varepsilon/2. \tag{2.2}$$

Define

$$\varphi_0 = \varphi\cdot \mathbb{1}_{\mathbb{N}\setminus P} + e^{-\arg(x)i}\cdot \mathbb{1}_P \in S_{\ell_\infty}, \tag{2.3}$$

$$x_0 = M^{-1}x\cdot \mathbb{1}_P\in S_{\ell_1}. \tag{2.4}$$

On one hand, we can compute

$$\begin{aligned}\|x-x_0\| &\overset{(2.4)}{=} \|x-M^{-1}x\cdot \mathbb{1}_P\| = (M^{-1}-1)\|x\cdot \mathbb{1}_P\| + \|x\cdot \mathbb{1}_{\mathbb{N}\setminus P}\| \\ &\overset{(2.2)}{=} (1-M)+\|x\cdot \mathbb{1}_{\mathbb{N}\setminus P}\| \overset{\|x\|\leq 1}{\leq} 2-2M \overset{(2.2)}{\leq} \varepsilon,\end{aligned}$$

and, since the support of x_0 is included in P (as a consequence of (2.4)), we deduce that

$$\begin{aligned}\varphi_0(x_0) &= \sum_{j\in P}(\varphi_0)_j(x_0)_j \overset{(2.3)}{=} \sum_{j\in P} e^{-\arg(x_j)i}(x_0)_j \overset{(2.4)}{=} \sum_{j\in P}|(x_0)_j| \\ &= \|x_0\| = 1,\end{aligned}$$

that is, $(x_0,\varphi_0)\in\Pi(\ell_1)$.

On the other hand, using

$$|z-1|\leq\sqrt{2(1-\mathrm{Re}(z))}\quad \text{for every } z\in\mathbb{C} \text{ such that } |z|\leq 1, \tag{2.5}$$

we deduce

$$\begin{aligned}\|\varphi-\varphi_0\| &\overset{(2.3)}{=} \sup_{j\in P}\{|\varphi_j-(\varphi_0)_j|\} \overset{(2.3)}{=} \sup_{j\in P}\{|\varphi_j-e^{-\arg(x_j)i}|\}\\ &= \sup_{j\in P}\{|e^{\arg(x_j)i}\varphi_j-1|\} \overset{(2.5)}{\le} \sup_{j\in P}\Big\{\sqrt{2-2\operatorname{Re}(e^{\arg(x_j)i}\varphi_j)}\Big\}\\ &\le \sqrt{2-2(1-\varepsilon^2/2)}=\varepsilon. \ \blacksquare\end{aligned}$$

An immediate consequence of Theorem 2.4 is the following version of the Bishop–Phelps–Bollobás theorem for $\ell_1(\mathbb{C})$.

Corollary 2.5. *Let $0<\varepsilon<1$ and $(x,\varphi)\in B_{\ell_1}\times B_{\ell_\infty}$ be such that $|\varphi(x)|\ge 1-\varepsilon^3/4$. Then there is $(x_0,\varphi_0)\in S_{\ell_1}\times S_{\ell_\infty}$ such that $\|x-x_0\|\le\varepsilon$, $\|\varphi-\varphi_0\|\le\varepsilon$ and $|\varphi_0(x_0)|=1$.*

Proof. Apply Theorem 2.4 to the pair $(e^{-\arg(\varphi(x))i}x,\varphi)$ obtaining (z_0,φ_0) belonging to $\Pi(\ell_1)$ such that $\|e^{-\arg(\varphi(x))i}x-z_0\|\le\varepsilon$ and $\|\varphi-\varphi_0\|\le\varepsilon$. If we set $x_0:=e^{\arg(\varphi(x))i}\,z_0$, the pair (x_0,φ_0) satisfies the conclusion. ■

2.2. Second constructive version. Given a pair (x,φ) and $0<\varepsilon<1$, Theorem 2.4 ensures the existence of a pair (x_0,φ_0) (defined by (2.4) and (2.3)) satisfying the conclusions of the Bishop–Phelps–Bollobás theorem. However, φ_0 depends on x, in fact, on $\arg(x)$. In order to prove Theorem 3.1 we will need a functional φ_0 depending only on the given ε and φ. So, we present the following result.

Theorem 2.6. *Let $(x,\varphi)\in B_{\ell_1}\times B_{\ell_\infty}$ and $0<\varepsilon<1$ be such that $\operatorname{Re}(\varphi(x))\ge 1-\varepsilon^3/60$. Then there exists $(x_0,\varphi_0)\in\Pi(\ell_1)$ such that $\|x-x_0\|\le\varepsilon$ and $\|\varphi-\varphi_0\|\le\varepsilon$. Moreover, the functional φ_0 can be defined as*

$$\varphi_0=\varphi\cdot\mathbb{1}_{\mathbb{N}\setminus\mathcal{A}_\varphi(\varepsilon^2/20)}+e^{\arg(\varphi)i}\cdot\mathbb{1}_{\mathcal{A}_\varphi(\varepsilon^2/20)}. \tag{2.6}$$

Proof. Consider the isometry $S\colon\ell_1\to\ell_1$ defined by

$$\langle e_j^*,Sy\rangle=e^{\arg(\varphi_j)i}\,y_j \quad \text{for } y\in\ell_1 \text{ and } j\in\mathbb{N}. \tag{2.7}$$

Set $\widetilde{x}=Sx$ and $\widetilde{\varphi}=\varphi\circ S^{-1}$. Then it is clear that the pair $(\widetilde{x},\widetilde{\varphi})$ is in $B_{\ell_1}\times B_{\ell_\infty}$, that $\operatorname{Re}(\widetilde{\varphi}(\widetilde{x}))\ge 1-\varepsilon^3/60$ and that $\widetilde{\varphi}=(|\varphi_j|)_{j\in\mathbb{N}}$ is positive. Denote $A=\mathcal{A}_{\widetilde{\varphi}}(r)$ and $P=\mathcal{P}_{(\widetilde{x},\widetilde{\varphi})}(r)$ (see definitions (1.3) and (1.4)) where $r:=\varepsilon^2/20$. Define

$$\widehat{\varphi}=\widetilde{\varphi}\cdot\mathbb{1}_{\mathbb{N}\setminus A}+\mathbb{1}_A\in S_{\ell_\infty}, \tag{2.8}$$

$$\widehat{x}=M^{-1}\operatorname{Re}(\widetilde{x})\cdot\mathbb{1}_P\in S_{\ell_1}, \tag{2.9}$$

where $M:=\|\operatorname{Re}(\widetilde{x})\cdot\mathbb{1}_P\|$. Applying Lemma 2.3 with $\delta=\varepsilon^3/60$ and r gives $M\ge 1-\varepsilon/3$. In particular, this means that P, and thus A, is non-empty.

We can compute that

$$\|\widetilde{\varphi}-\widehat{\varphi}\| \overset{(2.8)}{=} \sup_{j\in A}\{|\widetilde{\varphi}_j-\widehat{\varphi}_j|\} \overset{(2.8)}{=} \sup_{j\in A}\{|\widetilde{\varphi}_j-1|\} = \sup_{j\in A}\{(1-\widetilde{\varphi}_j)\} \overset{(1.3)}{\leq} r \leq \varepsilon, \tag{2.10}$$

and, since by (1.4) and (2.9) the support of $\widehat{x}$ is $P \subset A$ (which, in particular, implies that $\widehat{x}_j > 0$ for $j \in P$), we deduce that

$$\widehat{\varphi}(\widehat{x}) = \sum_{j\in P}\widehat{\varphi}_j\widehat{x}_j \overset{(2.8)}{=} \sum_{j\in P}\widehat{x}_j = \sum_{j\in P}|\widehat{x}_j| = 1, \tag{2.11}$$

that is, $(\widehat{x},\widehat{\varphi}) \in \Pi(\ell_1)$.

In order to show that $\|\widetilde{x}-\widehat{x}\| \leq \varepsilon$, observe first that

$$\|\widetilde{x}\cdot \mathbb{1}_P\| = \sum_{j\in P}|\widetilde{x}_j| \geq \sum_{j\in P}|\mathrm{Re}(\widetilde{x}_j)| = M \geq 1-\varepsilon/3, \tag{2.12}$$

from which

$$\begin{aligned}\|\widetilde{x}-\widehat{x}\| &\overset{(2.9)}{=} \|\widetilde{x}-M^{-1}\,\mathrm{Re}(\widetilde{x})\cdot \mathbb{1}_P\| \\ &= \|\widetilde{x}\cdot \mathbb{1}_{\mathbb{N}\setminus P}\| + \|(\widetilde{x}-M^{-1}\,\mathrm{Re}(\widetilde{x}))\cdot \mathbb{1}_P\| \\ &\overset{(2.12)}{\leq} \varepsilon/3 + \|(\widetilde{x}-M^{-1}\,\mathrm{Re}(\widetilde{x}))\cdot \mathbb{1}_P\|.\end{aligned} \tag{2.13}$$

We need a bit more care to estimate the last term in (2.13). From the very definition of P, we know that for every $j \in P$,

$$|\widetilde{x}_j| \leq (1-r)^{-1}\widetilde{\varphi}_j\,\mathrm{Re}(\widetilde{x}_j). \tag{2.14}$$

Therefore,

$$\begin{aligned}\|(\widetilde{x}-\mathrm{Re}(\widetilde{x}))\cdot \mathbb{1}_P\| &= \sum_{j\in P}|\widetilde{x}_j-\mathrm{Re}(\widetilde{x}_j)| = \sum_{j\in P}|\mathrm{Im}(\widetilde{x}_j)| \\ &= \sum_{j\in P}\sqrt{|\widetilde{x}_j|^2-\mathrm{Re}(\widetilde{x}_j)^2} \\ &\overset{(2.14)}{\leq} \sum_{j\in P}|\mathrm{Re}(\widetilde{x}_j)|\sqrt{(1-r)^{-2}-1} \\ &\leq \|\widetilde{x}\|\sqrt{(1-r)^{-2}-1} \overset{r=\varepsilon^2/20}{\leq} \varepsilon/3,\end{aligned} \tag{2.15}$$

which implies that

(2.16)

$$\begin{aligned}\|(\widetilde{x}-M^{-1}\operatorname{Re}(\widetilde{x}))\cdot\mathbb{1}_P\| &\le \|(\widetilde{x}-\operatorname{Re}(\widetilde{x}))\cdot\mathbb{1}_P\|+\|(1-M^{-1})\operatorname{Re}(\widetilde{x})\cdot\mathbb{1}_P\|\\ &\overset{(2.15)}{\le} \varepsilon/3+(M^{-1}-1)\|\operatorname{Re}(\widetilde{x})\cdot\mathbb{1}_P\|\\ &=\varepsilon/3+(1-M)\le 2\varepsilon/3.\end{aligned}$$

Putting together (2.13) and (2.16), one obtains

$$\|\widetilde{x}-\widehat{x}\|\le\varepsilon/3+\|(\widetilde{x}-M^{-1}\operatorname{Re}(\widetilde{x}))\cdot\mathbb{1}_P\|\le\varepsilon, \tag{2.17}$$

which finishes the core of the proof.

Now, we define

$$x_0=S^{-1}\widehat{x}\quad\text{and}\quad\varphi_0=S^*(\widehat{\varphi})=\widehat{\varphi}\circ S, \tag{2.18}$$

which by (2.11) gives $\varphi_0(x_0)=\widehat{\varphi}(\widehat{x})=1$. Since S and S^* are isometries, we deduce from (2.10), (2.17), (2.18) and the definition of $\widetilde{x}$ and $\widetilde{\varphi}$ that

$$\|x-x_0\|\le\varepsilon,\qquad\|\varphi-\varphi_0\|\le\varepsilon.$$

Therefore, (x_0,φ_0) is the pair in $\Pi(\ell_1)$ we were looking for.

Bearing in mind (2.18), one computes

$$(\varphi_0)_j=\varphi_0(e_j)\overset{(2.18)}{=}\widehat{\varphi}(Se_j)\overset{(2.7)}{=}\widehat{\varphi}(e^{\arg(\varphi_j)i}\,e_j)=e^{\arg(\varphi_j)i}\,\widehat{\varphi}_j,$$

which together with (2.8) implies that $\varphi_0=\varphi\cdot\mathbb{1}_{\mathbb{N}\setminus A}+e^{\arg(\varphi)i}\cdot\mathbb{1}_A$. Finally, in view of $A=\mathcal{A}_{\widetilde{\varphi}}(r)=\mathcal{A}_{\varphi}(r)$, the validity of (2.6) has been shown. ∎

Remark 2.7. Observe that the function φ_0 provided by Theorem 2.6 and defined by (2.6) only depends on ε and φ itself, as well as satisfies $\pi_1(\varphi)\subset\pi_1(\varphi_0)$.

3. BPB property for numerical radius in $\ell_1(\mathbb{C})$. As a consequence of Theorems 2.4 and 2.6 we show that ℓ_1 has the Bishop–Phelps–Bollobás property for numerical radius.

Theorem 3.1. *Let $T\in S_{\mathfrak{L}(\ell_1)}$, $0<\varepsilon<1$ and $(x,\varphi)\in\Pi(\ell_1)$ be such that $\varphi(Tx)\ge 1-(\varepsilon/9)^{9/2}$. Then there exist $T_0\in S_{\mathfrak{L}(\ell_1)}$ and $(x_0,\varphi_0)\in\Pi(\ell_1)$ such that*

$$\|T-T_0\|\le\varepsilon,\quad\|x-x_0\|\le\varepsilon,\quad\|\varphi-\varphi_0\|\le\varepsilon,\quad\varphi_0(T_0x_0)=1. \tag{3.1}$$

Proof. First of all, fix $\mu:=\sqrt{\varepsilon^3/240}$. Using a suitable isometry, we can assume that x is positive. In particular, by Lemma 2.1 and the definition of $\mathcal{N}_{x,\varphi}$ in (1.2), we can assume that $\varphi_j=1$ for $j\in\operatorname{supp}(x)$. Since $\mu^3/4\ge(\varepsilon/9)^{9/2}$, Theorem 2.4 can be applied to the pair $(x,T^*\varphi)\in B_{\ell_1}\times B_{\ell_\infty}$ and μ instead of ε giving $x_0\in\pi_1(\varphi)$ such that $\|x-x_0\|\le\mu\le\varepsilon$. Moreover, by (2.1) we know that

$$x_0 = \|x \cdot \mathbb{1}_P\|^{-1} \cdot x \cdot \mathbb{1}_P, \tag{3.2}$$

where the non-empty set P is defined by

$$P := \mathcal{P}_{(x,T^*\varphi)}(\mu^2/2) = \{j \in \operatorname{supp}(x) \colon \operatorname{Re}(T^*\varphi(e_j)) \geq 1 - \mu^2/2\}. \tag{3.3}$$

In particular, x_0 is positive.

Since $\mu^2/2 = (\varepsilon/2)^3/60$, for each $j \in P$ we can apply Theorem 2.6 to the pair $(e^{-\arg(\varphi(Te_j))i}Te_j, \varphi)$ and $\varepsilon/2$ to find $(z_j, \varphi_0) \in \Pi(\ell_1)$ such that

$$\|Te_j - a_j z_j\| \leq \varepsilon/2, \qquad \|\varphi - \varphi_0\| \leq \varepsilon/2$$

and $\Pi_1(\varphi) \subset \Pi_1(\varphi_0)$ (see Remark 2.7), where $a_j = e^{\arg(\varphi(Te_j))i}$. Observe that φ_0 can be chosen independently of $j \in P$ and by (2.6) explicitly written as

$$\varphi_0 = \varphi \cdot \mathbb{1}_{\mathbb{N}\setminus\mathcal{A}_\varphi(\varepsilon^2/80)} + e^{\arg(\varphi)i} \cdot \mathbb{1}_{\mathcal{A}_\varphi(\varepsilon^2/80)}. \tag{3.4}$$

Define T_0 as the unique operator in $\mathfrak{L}(\ell_1)$ such that $T_0 e_i = Te_i$ for $i \notin P$ and $T_0 e_j = z_j$ for $j \in P$. Equivalently,

$$T_0 x = \mathbb{1}_{\mathbb{N}\setminus P} \cdot Tx + \sum_{j\in P} e_j^*(x) z_j \quad \text{for } x \in \ell_1. \tag{3.5}$$

It is clear from (3.5) that

$$\|T_0\| = \sup_{n\in\mathbb{N}} \{\|T_0 e_n\|\} = \max\Big\{\sup_{j\notin P} \{\|Te_j\|\}, \sup_{j\in P} \{\|z_j\|\}\Big\} = 1.$$

Given $j \in P$, the identity (3.3) ensures that $\operatorname{Re}(\varphi(Te_j)) \geq 1 - \mu^2/2$. Using again the general fact (2.5), we deduce that $|a_j - 1| \leq \mu \leq \varepsilon/2$.

Therefore,

$$\begin{aligned}\|T - T_0\| &= \sup_{n\in\mathbb{N}} \{\|Te_n - T_0 e_n\|\} = \sup_{j\in P} \{\|Te_j - z_j\|\} \\ &\leq \sup_{j\in P} \{\|Te_j - a_j z_j\|\} + \sup_{j\in P} \{\|a_j z_j - z_j\|\} \\ &\leq \varepsilon/2 + \sup_{j\in P} \{|a_j - 1|\} \leq \varepsilon.\end{aligned}$$

Since $x_0 \in \pi_1(\varphi)$ and $\pi_1(\varphi) \subset \pi_1(\varphi_0)$, we deduce that (x_0, φ_0) belongs to $\Pi(\ell_1)$. It remains to show that $\varphi_0(T_0 x_0) = 1$ to prove the validity of (3.1). But, since x_0 is positive, we obtain

$$\begin{aligned}\varphi_0(T_0 x_0) &\overset{(3.5)}{=} \sum_{j\in P} (x_0)_j \varphi_0(z_j) + \sum_{j\notin P} (x_0)_j \varphi_0(Te_j) \\ &\overset{(3.2)}{=} \sum_{j\in P} (x_0)_j = \sum_{j\in P} |(x_0)_j| = \|x_0\| = 1,\end{aligned}$$

and the proof is complete. ■

Remark 3.2. We cannot replace the condition $(x,\varphi)\in\Pi(\ell_1)$ in Theorem 3.1 by the more general $(x,\varphi)\in B_{\ell_1}\times B_{\ell_\infty}$. Indeed, consider the operator $T\colon \ell_1\to\ell_1$ defined by $Te_j=e_j$ for $j\geq 2$ and $Te_1=e_2$. Take $(e_1,e_2^*)\in B_{\ell_1}\times B_{\ell_\infty}$, $T_0\in\mathfrak{L}(\ell_1)$, and $(x,\varphi)\in B_{\ell_1}\times B_{\ell_\infty}$ such that $\|T-T_0\|\leq\varepsilon$, $\|e_1-x\|\leq\varepsilon$, and $\|e_2^*-\varphi\|\leq\varepsilon$. Then

$$|\varphi(x)|\leq|\varphi(x)-e_2^*(x)|+|e_2^*(x)-e_2^*(e_1)|+|e_2^*(e_1)|\leq 2\varepsilon,$$

which implies that (x,φ) cannot be in $\Pi(\ell_1)$.

Corollary 3.3. *The Banach space ℓ_1 has the Bishop–Phelps–Bollobás property for numerical radius.*

Proof. Consider $T\in\mathfrak{L}(\ell_1)$ with $\nu(T)=1$ and $0<\varepsilon<1$. Take a pair $(x,\varphi)\in\Pi(\ell_1)$ such that $|\varphi(Tx)|\geq 1-(\varepsilon/9)^{9/2}$. In fact, we can assume that $\varphi(Tx)\geq 1-(\varepsilon/9)^{9/2}$; otherwise, we proceed with $\widetilde{T}=e^{-\arg(\varphi(Tx))i}\,T$. Then Theorem 3.1 gives the existence of an operator $T_0\in S_{\mathfrak{L}(\ell_1)}$ and a pair $(x_0,\varphi_0)\in\Pi(\ell_1)$ that satisfy the conditions in (3.1), which are precisely the requirements (1.1) in Definition 1.2. ∎

Corollary 3.4 ([Car85b]). *The set* NRA(ℓ_1) *is dense in* $\mathfrak{L}(\ell_1)$.

4. BPB property for numerical radius in $c_0(\mathbb{C})$. Theorem 3.1 allows us to show that c_0 has the Bishop–Phelps–Bollobás property for numerical radius as well. Indeed, we rely on the fact that our constructions in ℓ_1 can be dualized.

Theorem 4.1. *Let $T\in S_{\mathfrak{L}(c_0)}$, $0<\varepsilon<1$ and $(x,\varphi)\in\Pi(c_0)$ be such that $|\varphi(Tx)|\geq 1-(\varepsilon/9)^{9/2}$. Then there exist $S\in S_{\mathfrak{L}(c_0)}$ and $(x_0,\varphi_0)\in\Pi(c_0)$ such that*

$$\|T-S\|\leq\varepsilon,\quad \|x-x_0\|\leq\varepsilon,\quad \|\varphi-\varphi_0\|\leq\varepsilon,\quad |\varphi_0(Sx_0)|=1.$$

Proof. Throughout this proof we identify the elements in c_0 with their images in ℓ_∞ under the natural embedding $c_0\to\ell_\infty$. The adjoint operator of T, $T^*\colon\ell_1\to\ell_1$, satisfies

$$|x(T^*\varphi)|=|T^*(\varphi)(x)|=|\varphi(Tx)|\geq 1-(\varepsilon/9)^{9/2}.$$

Without loss of generality, we can assume that $x(T^*\varphi)\geq 1-(\varepsilon/9)^{9/2}$. Otherwise, employing the techniques from the proof of Corollary 3.3, define the operator $\widetilde{T}=e^{-\arg(x(T^*\varphi))i}\,T^*$ and proceed with the proof for $x(\widetilde{T}\varphi)=|x(T^*\varphi)|$.

By Theorem 3.1, there exist $T_0\in\mathfrak{L}(\ell_1)$ with $\|T_0\|=1$ and $(\varphi_0,x_0)\in\Pi(\ell_1)$ such that

$$\|T^*-T_0\|\leq\varepsilon,\quad \|\varphi-\varphi_0\|\leq\varepsilon,\quad \|x-x_0\|\leq\varepsilon$$

and $x_0(T_0\varphi_0)=1$.

We assert that (x_0, φ_0) is the pair we are looking for. To show this, we will reexamine the proof of Theorem 3.1 to check how x_0, φ_0 and T_0 are defined. Indeed, from (3.3), (3.2), (3.4) and (3.5) we have respectively

$$\begin{aligned} P &= \mathcal{P}_{(\varphi, T^{**}x)}(\varepsilon^3/480), \\ \varphi_0 &= \|\varphi \cdot \mathbb{1}_P\|^{-1} \cdot \varphi \cdot \mathbb{1}_P, \\ x_0 &= x \cdot \mathbb{1}_{\mathbb{N} \setminus A_x(\varepsilon^2/80)} + e^{\arg(x)i} \cdot \mathbb{1}_{A_x(\varepsilon^2/80)}, \\ T_0 x &= \mathbb{1}_{\mathbb{N} \setminus P} \cdot Tx + \sum_{j \in P} e_j^*(x) z_j \quad \text{for } x \in \ell_1, \end{aligned} \tag{4.1}$$

where $\{z_j\}_{j \in P} \subset \pi_1(\varphi_0)$.

Note that $\mathcal{A}_x(\varepsilon^2/80) = \{j \in \mathbb{N} \colon |x_j| \geq 1 - \varepsilon^2/80\}$ and $x \in c_0$. Thus, $A_x(\varepsilon^2/80)$ is finite, which, by (4.1), implies that $x_0 \in c_0$.

We shall show that T_0 is an adjoint operator and thus there exists $S \in \mathfrak{L}(c_0)$ such that $S^* = T_0$. It will be enough to show that $T_0^*|_{c_0} \subset c_0$. Set $t_{ij} = \langle e_i, T(e_j) \rangle$ for $i, j \in \mathbb{N}$. Fix $i \in \mathbb{N}$; then for $j \in \mathbb{N}$,

$$\langle e_j, T_0^*(e_i) \rangle = \begin{cases} t_{ji} & \text{if } j \notin P, \\ (z_j)_i & \text{if } j \in P. \end{cases}$$

Since $x \in c_0$, $T^{**}x$ belongs to c_0, which implies that P is finite. Accordingly, only finitely many terms of the form $\langle e_j, T_0^*(e_i) \rangle$ differ from the corresponding t_{ji}. On the other hand, since T belongs to $\mathfrak{L}(c_0)$, we have $\lim_j |t_{ji}| = 0$. Therefore, $|\langle e_j, T_0^*(e_i) \rangle| \to 0$ as $j \to \infty$. This implies that $T_0^* e_i \in c_0$ and, since $i \in \mathbb{N}$ is arbitrarily chosen, we deduce that $T_0^*|_{c_0} \subset c_0$.

Hence the operator $S = T_0^*|_{c_0} \in \mathfrak{L}(c_0)$ and the pair $(x_0, \varphi_0) \in \Pi(c_0)$ satisfy

$$\varphi_0(Sx_0) = S^*\varphi_0(x_0) = x_0(S^*\varphi_0) = x_0(T_0\varphi_0) = 1,$$

and

$$\|S - T\| = \|(S - T)^*\| = \|S^* - T^*\| = \|T_0 - T^*\| \leq \varepsilon,$$

which finishes the proof. ∎

Theorem 4.1 implies the following two corollaries.

Corollary 4.2. *The Banach space c_0 has the Bishop–Phelps–Bollobás property for numerical radius.*

Corollary 4.3 ([Car85b]). *The set* $\mathrm{NRA}(c_0)$ *is dense in* $\mathfrak{L}(c_0)$.

5. Generalizations and remarks. All the results of Sections 2–4 were stated and proved for the Banach spaces $\ell_1(\mathbb{C})$ or $c_0(\mathbb{C})$. However, a glance at their proofs shows that they remain valid for $\ell_1(\mathbb{R})$ and $c_0(\mathbb{R})$, with shorter proofs and better estimates. More generally, given a non-empty set Γ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, these results are, after suitable adjustments, still valid for $\ell_1(\Gamma, \mathbb{K})$ and $c_0(\Gamma, \mathbb{K})$. The spaces $\ell_1(\Gamma, \mathbb{K})$ and $c_0(\Gamma, \mathbb{K})$ are, respectively, the ℓ_1-sum and the c_0-sum of Γ copies of the field $\mathbb{K}$. Note that in particular $\ell_1(\mathbb{N}, \mathbb{K}) = \ell_1(\mathbb{K})$.

The Banach space $c_0(\Gamma, \mathbb{K})$ is a predual of $\ell_1(\Gamma, \mathbb{K})$. Observe that both $c_0(\Gamma, \mathbb{K})$ and $\ell_1(\Gamma, \mathbb{K})$ have numerical index 1. Previous considerations imply that both also have the BPB property for numerical radius. The ω^* topology of $\ell_1(\Gamma, \mathbb{K})$ below is the topology induced on $\ell_1(\Gamma, \mathbb{K})$ by pointwise convergence on elements of $c_0(\Gamma, \mathbb{K})$.

On the other hand, the proof of Theorem 4.1 shows that in Theorem 3.1 we proved more than was stated. Indeed, putting together Theorem 3.1, the ideas on duality in the proof of Theorem 4.1 and the considerations above, one easily proves the following theorem.

Theorem 5.1. *Let* $T \in S_{\mathfrak{L}(\ell_1(\Gamma,\mathbb{K}))}$, $0 < \varepsilon < 1$ *and* $(x, \varphi) \in \Pi(\ell_1(\Gamma, \mathbb{K}))$ *be such that* $|\varphi(Tx)| \geq 1 - (\varepsilon/9)^{9/2}$. *Then there exist* $T_0 \in S_{\mathfrak{L}(\ell_1(\Gamma,\mathbb{K}))}$ *and* $(x_0, \varphi_0) \in \Pi(\ell_1(\Gamma, \mathbb{K}))$ *such that*

$$\|T - T_0\| \leq \varepsilon, \quad \|x - x_0\| \leq \varepsilon, \quad \|\varphi - \varphi_0\| \leq \varepsilon, \quad |\varphi_0(T_0 x_0)| = 1.$$

Moreover, if T *is* ω^*-ω^*-*continuous and* φ *is* ω^*-*continuous, then* T_0 *and* φ_0 *are* ω^*-ω^*-*continuous and* ω^*-*continuous, respectively.*

Below are two consequences of Theorem 5.1.

Theorem 5.2. *The Banach space* $\ell_1(\Gamma, \mathbb{K})$ *has the BPB property for numerical radius.*

Theorem 5.3. *The Banach space* $c_0(\Gamma, \mathbb{K})$ *has the BPB property for numerical radius.*

Proof. Fix $0 < \varepsilon < 1$, $\delta \leq (\varepsilon/9)^{9/2}$, $T \in S_{\mathfrak{L}(c_0(\Gamma,\mathbb{K}))}$ and $(x, x^*) \in \Pi(c_0(\Gamma, \mathbb{K}))$ such that $|x^*(Tx)| \geq 1 - \delta$. Applying Theorem 5.1 to the ω^*-ω^*-continuous operator $T^* \in S_{\mathfrak{L}(\ell_1(\Gamma,\mathbb{K}))}$, the pair (x^*, x) and ε gives a new $T_0 \in S_{\mathfrak{L}(c_0(\Gamma,\mathbb{K}))}$ and a new pair $(x_0^*, x_0^{**}) \in \Pi(\ell_1(\Gamma, \mathbb{K}))$ satisfying

$$\|T^* - T_0^*\| \leq \varepsilon, \quad \|x - x_0^{**}\| \leq \varepsilon, \quad \|x^* - x_0^*\| \leq \varepsilon, \quad |x_0^{**}(T_0^* x_0^*)| = 1. \tag{5.1}$$

Moreover, x_0^{**} is ω^*-continuous, so we can identify it with some $x_0 \in S_{c_0(\Gamma,\mathbb{K})}$. Therefore, conditions in (5.1) become

$$\big\|T - T_0\big\| \leq \varepsilon, \quad \|x - x_0\| \leq \varepsilon, \quad \|x^* - x_0^*\| \leq \varepsilon, \quad |x_0^*(T_0 x_0)| = 1.$$

which are the requirements (1.1) in Definition 1.2. Consequently, $c_0(\Gamma, \mathbb{K})$ has the Bishop–Phelps–Bollobás property for numerical radius. ■

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