The Einstein Constraint Equations: An Introduction

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Overview

Goals

- Understand how to derive the Einstein Constraint Equations.
- Explore the role of the constraints in the initial-value problem for Einstein's equation, and thus appreciate why it is of interest to constraint solutions to the constraint equations.
- Introduce (one or more, as time permits) approaches for solving the Einstein Constraint Equations.
- As time permits, explore the geometry of solutions of the constraints.

Outline

Part I: The basics.

Introduction

- Derive the Einstein constraint equations.
- Initial data for the Einstein equation.
- Hamiltonian formulation; dynamics.
- Constraints operator and its linearization.

Modeling Isolated Systems

- Asymptotically flat solutions (AF) of the constraints.
- The energy and momenta of AF solutions, and the relation to the constraints operator.
- Statement of Positive Mass Theorem.

Outline

Part I: The basics (continued)

Constructing Solutions: Conformal Techniques.

- Conformal deformations; applications to AF solutions.
- The basic framework of the conformal method of Lichnerowicz and York: goal is to effectively parametrize the moduli space of solutions.
- CMC case.

Part II. Topics, as time permits.

- Discussion of non-CMC case, recent works.
- Constructing New Solutions from Old: Gluing methods and a variety of applications.
- Geometry of the constraints: Riemannian case of Positive Mass Theorem.

Curvature conventions

We define the curvature for an affine connection ∇ as $R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

We use the index convections (and the Einstein summation convention)

$$R = R^{\ell}_{ijk} \frac{\partial}{\partial x^{\ell}} \otimes dx^i \otimes dx^j \otimes dx^k$$

and with metric $g = \langle \cdot, \cdot \rangle$, $R_{ijk\ell} = \langle R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}), \frac{\partial}{\partial x^{\ell}} \rangle = R^m_{ijk} g_{m\ell}$.
Ricci curvature: $R_{jk} = R^{\ell}_{\ell jk}$.

Scalar curvature: $R(g) = g^{ij} R_{ij}$.

The Second Fundamental Form

We will study the geometry of space-like hypersurfaces M inside a space-time $(\mathcal{S}, \bar{g})$, of dimension $\dim(\mathcal{S}) = m + 1$, where we will for simplicity often take $m = 3$.

For the moment, let $M \subset (\mathcal{S}, \bar{g})$ be an embedded submanifold, where $\bar{g} = \langle \cdot, \cdot \rangle$ is non-degenerate (Riemannian or Lorentzian, say).

We assume the metric $\bar{g}$ induces a non-degenerate metric g (the First Fundamental Form) on M. For the induced metric g, the Levi-Civita connection ∇ is related to the connection $\overline{\nabla}$ of $\bar{g}$ by

$$\overline{\nabla}_X Y = (\overline{\nabla}_X Y)^{\mathrm{Tan}} + (\overline{\nabla}_X Y)^{\mathrm{Nor}} = \nabla_X Y + \mathbb{II}(X, Y).$$

For X and Y tangent to M, so that $\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y]$ is also tangent to M, so that $\mathbb{II}(X, Y) = \mathbb{II}(Y, X)$.
From this we can then see that $\mathbb{II}(X, Y)$ is tensorial in X and Y. The symmetric tensor $\mathbb{II}$ is called the vector-valued second fundamental form.

The Second Fundamental Form

In the case M is a hypersurface ($\dim(M) = m = \dim(\mathcal{S}) - 1$), we let n be a (local) unit normal field to M.

From $\langle \overline{\nabla}_X n, n \rangle = 0$, we see $\overline{\nabla}_X n$ is tangent to M.

We also note that for X and Y tangent to M, $\langle \overline{\nabla}_X Y, n \rangle = -\langle Y, \overline{\nabla}_X n \rangle$.

Scalar-valued second fundamental form

We define $\hat{K}$ by

$\hat{K}(X, Y)n = \mathbb{II}(X, Y) = \langle n, n \rangle \langle \mathbb{II}(X, Y), n \rangle n = \langle n, n \rangle \langle \overline{\nabla}_X Y, n \rangle n.$

We let $K(X, Y) = \langle -\overline{\nabla}_X n, Y \rangle = \langle \overline{\nabla}_X Y, n \rangle = \langle n, n \rangle \hat{K}(X, Y)$.

We refer to K or $\hat{K}$ as the scalar-valued second fundamental form.

In case $(\mathcal{S}, \bar{g})$ is Lorentzian, $\langle n, n \rangle = -1$ and so

$$\hat{K}(X, Y) = \langle \overline{\nabla}_X n, Y \rangle.$$

The Gauss Equation

There are fundamental equations related the curvature tensor $\bar{R}$ of $\bar{g}$, R of g and the second fundamental form $\mathbb{II}$ (or $\hat{K} = \langle n, n \rangle K$).
The Gauss equation relates the curvature tensor on the submanifold M to that of the ambient manifold, with the difference measured using the second fundamental form.

The Gauss Equation

For X, Y, Z, W tangent to M,

$$\begin{aligned}\langle R(X,Y,Z), W\rangle = &\langle \bar{R}(X,Y,Z), W\rangle \\ &+ \langle \mathbb{II}(X,W), \mathbb{II}(Y,Z)\rangle - \langle \mathbb{II}(X,Z), \mathbb{II}(Y,W)\rangle\end{aligned}$$

Hypersurface case:

$$\begin{aligned}\langle R(X,Y,Z), W\rangle = &\langle \bar{R}(X,Y,Z), W\rangle \\ &+ \langle n, n\rangle \Big[K(X,W)K(Y,Z) - K(X,Z)K(Y,W)\Big]\end{aligned}$$

The Gauss Equation

Proof: The proof proceeds by decomposing terms in $\bar{R}(X, Y, Z)$ into tangential and normal components, and discarding normal terms that disappear in the inner product with vectors tangent to M:

$$\begin{aligned}\langle \bar{R}(X, Y, Z), W\rangle &= \langle \overline{\nabla}_X(\nabla_Y Z + \mathbb{II}(Y, Z)), W\rangle \\ &\quad - \langle \overline{\nabla}_Y(\nabla_X Z + \mathbb{II}(X, Z)), W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle\end{aligned}$$

and then again (similarly for the other term)

$$\overline{\nabla}_X(\nabla_Y Z + \mathbb{II}(Y, Z)) = \nabla_X \nabla_Y Z + \mathbb{II}(X, \nabla_Y Z) + \overline{\nabla}_X(\mathbb{II}(Y, Z)).$$

Upon inner product with W (tangent to M), we obtain

$$\begin{aligned}\langle \overline{\nabla}_X(\nabla_Y Z + \mathbb{II}(Y, Z)), W\rangle &= \langle \nabla_X \nabla_Y Z, W\rangle + \langle \overline{\nabla}_X(\mathbb{II}(Y, Z)), W\rangle \\ &= \langle \nabla_X \nabla_Y Z, W\rangle - \langle \mathbb{II}(Y, Z), \overline{\nabla}_X W\rangle \\ &= \langle \nabla_X \nabla_Y Z, W\rangle - \langle \mathbb{II}(Y, Z), \mathbb{II}(X, W)\rangle.\end{aligned}$$

The Codazzi Equation

The Codazzi equation involves the normal component $\bar{R}^{\perp}(X,Y,Z)$ of $\bar{R}(X,Y,Z)$, whereas the Gauss equation involved the tangential component of this vector.

For simplicity, we derive it only in the hypersurface case. Now, $\bar{R}(X,Y,Z) = \overline{\nabla}_X(\nabla_Y Z + \hat{K}(Y,Z)n) - \overline{\nabla}_Y(\nabla_X Z + \hat{K}(X,Z)n) - \overline{\nabla}_{[X,Y]}Z$, the normal component is just (using $\overline{\nabla}_X n$ is tangential to M)

$$\begin{aligned}\bar{R}^{\perp}(X,Y,Z) &= \mathbb{II}(X, \nabla_Y Z) + \left[\overline{\nabla}_X(\hat{K}(Y,Z))\right]n - \mathbb{II}(Y, \nabla_X Z) \\ &\quad - \left[\overline{\nabla}_Y(\hat{K}(X,Z))\right]n - \left(\overline{\nabla}_{\nabla_X Y}Z - \overline{\nabla}_{\nabla_Y X}Z\right)^{\perp} \\ &= \Big[\hat{K}(X,\nabla_Y Z) + \nabla_X(\hat{K}(Y,Z)) - \hat{K}(Y,\nabla_X Z) \\ &\quad - \nabla_Y(\hat{K}(X,Z)) - \hat{K}(\nabla_X Y, Z) + \hat{K}(\nabla_Y X, Z)\Big]n\end{aligned}$$

since $(\overline{\nabla}_{\nabla_X Y}Z)^{\perp} = \mathbb{II}(\nabla_X Y, Z)$. By inspection, we see we have obtained

The Codazzi Equation

The Codazzi Equation

For X, Y, Z tangent to M,

$$\langle \bar{R}(X,Y,Z), n\rangle = \langle n, n\rangle\Big((\nabla_X \hat{K})(Y,Z) - (\nabla_Y \hat{K})(X,Z)\Big).$$

Index notation

In index notation, we might write the Gauss and Codazzi equations as follows, where i, j, k and ℓ indices are for components tangential to M, and the n index indicates that the vector n is placed in the indicated slot of the tensor. Gauss:

$R_{ijk\ell} = \bar{R}_{ijk\ell} + \langle n, n\rangle(\hat{K}_{i\ell}\hat{K}_{jk} - \hat{K}_{ik}\hat{K}_{j\ell}) = \bar{R}_{ijk\ell} + \langle n, n\rangle(K_{i\ell}K_{jk} - K_{ik}K_{j\ell})$.

Codazzi:

$\bar{R}_{ijkn} = \langle n, n\rangle(\hat{K}_{jk;i} - \hat{K}_{ik;j}) = K_{jk;i} - K_{ik;j}$

The Einstein Constraint Equations

Consider a hypersurface M in a space-time $(\mathcal{S}, \bar{g})$, so that the induced metric g on M is Riemannian. Let n be a (local) time-like unit normal field.

We suppose that $\bar{g}$ satisfies an Einstein equation ($\kappa = \frac{8\pi G}{c^4}$)

$$G_\Lambda(\bar{g}) = \mathrm{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} + \Lambda\bar{g} = \kappa T.$$

The Einstein constraint equations relate the first and second fundamental forms g and K on M.

They are obtained using the Gauss and Codazzi equations, together with the information about the ambient curvature contained in the Einstein equation.

The Einstein Constraint Equations

Let $\tilde{\mathbb{J}}^\nu = -T^{\mu\nu} n_\mu = -T^{\mu\nu} n^\beta \bar{g}_{\mu\beta}$; we write $\tilde{\mathbb{J}} = \rho n + \vec{J}$, with $\langle \vec{J}, n \rangle = 0$. We let J be the one-form dual to $\vec{J}$ using the metric g on M. If we let $E_1, \ldots, E_m$ be a basis for T_pM, with dual basis $\theta^1, \ldots, \theta^m$, then we can write $\vec{J} = J^\ell E_\ell$ and $J = J_i \theta^i$, with $J_i = J^\ell g_{i\ell}$. We let $E_0 = n$ to give a basis $\{E_\mu\}$ of $T_p\mathcal{S}$.

It's not hard to see (watch minus signs!) that $\rho = T(n, n)$; this is the energy density of matter fields as measured by an observed with four-velocity cn.

Furthemore, J is ($\pm c$ times) the corresponding observed momentum density one-form: since $n^0 = 1$, $\bar{g}_{00} = -1$, we have $n_0 = -1$, and so

$J^i = -T^{\mu i} n_\mu = T^{0i}$, and
$J_j = g_{j\ell} J^\ell = \bar{g}_{j\mu} T^{0\mu} = T^0_j = -\bar{g}_{00} T^0_j = -\bar{g}_{0\mu} T^\mu_j = -T_{0j} = -T_{j0}$.

Dominant energy condition: $\tilde{\mathbb{J}}$ is future-pointing causal: $\rho \geq |J|_g = \sqrt{J^i J_i}$.

The Einstein Constraint Equations

$$R(g) - \|K\|_g^2 + (\mathrm{tr}_g K)^2 = 2\kappa\rho + 2\Lambda$$
$$\mathrm{div}_g(K - (\mathrm{tr}_g K)g) = \mathrm{div}_g K - d(\mathrm{tr}_g(K)) = \kappa J.$$

In case $m = 3$, say, this is locally a system of four equations for (g, K), forming an underdetermined elliptic system.

Let $\Lambda = 0$ for now. Vacuum case would be $\rho = 0$, $J = 0$.

Maximal case: $\mathrm{tr}_g(K) = 0$: $R(g) = 2\kappa\rho + \|K\|_g^2 \geq 0$ (under DEC).

Time-symmetric case: $K = 0$: $R(g) = 2\kappa\rho \geq 0$ (under DEC) .

In this case, the scalar curvature is proportional to the observed local energy density.

Time-symmetric vacuum case: $R(g) = 0$.

We give the proof of the first constraint, *the Hamiltonian constraint*.

Proof: Let E_i be an orthonormal frame for T_pM. We use the Gauss equation (careful with the signs: $\langle n, n\rangle = -1$):

$$\sum_{i,j=1}^{m} \langle \bar{R}(E_i, E_j, E_j), E_i\rangle = \sum_{i,j=1}^{m} \Big[\langle R(E_i, E_j, E_j), E_i\rangle + K(E_j, E_j)K(E_i, E_i) - (K(E_i, E_j))^2\Big]$$
$$= R(g) - \|K\|_g^2 + (\mathrm{tr}_g(K))^2.$$

Again, $\langle n, n\rangle = -1$, so that

$\overline{\mathrm{Ric}}(E_j, E_j) = -\langle \bar{R}(n, E_j, E_j), n\rangle + \sum_{i=1}^{m}\langle \bar{R}(E_i, E_j, E_j), E_i\rangle$. Thus

$$\sum_{i,j=1}^{m} \langle \bar{R}(E_i, E_j, E_j), E_i\rangle = \overline{\mathrm{Ric}}(n, n) + \sum_{j=1}^{m} \overline{\mathrm{Ric}}(E_j, E_j).$$

Deriving the Einstein Constraint Equations

Recall the Einstein tensor: $G = \mathrm{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g}$.

$$
\begin{aligned}
\sum_{i,j=1}^{m} \langle \bar{R}(E_i, E_j, E_j), E_i \rangle &= \overline{\mathrm{Ric}}(n,n) + \sum_{j=1}^{m} \overline{\mathrm{Ric}}(E_j, E_j). \\
&= 2\overline{\mathrm{Ric}}(n,n) + \left(-\overline{\mathrm{Ric}}(n,n) + \sum_{j=1}^{m} \overline{\mathrm{Ric}}(E_j, E_j)\right) \\
&= 2\overline{\mathrm{Ric}}(n,n) + R(\bar{g}) \\
&= 2G(n,n) \\
&= 2(-\Lambda\bar{g} + \kappa T)(n,n) \\
&= 2\Lambda + 2\kappa\rho
\end{aligned}
$$

Emphasis: $2G(n,n) = R(g) - \|K\|_g^2 + (\mathrm{tr}_g(K))^2$.

Deriving the Einstein Constraint Equations

As for the momentum constraint, we employ the Codazzi equation, which we recall in index form: $\bar{R}_{ijkn} = K_{jk;i} - K_{ik;j}$.

Proof: Since $\bar{R}_{jinn} = 0$, we have

$$G_{in} = \bar{R}_{in} = \sum_{j=1}^{m} \langle \bar{R}(e_j, e_i, n), e_j \rangle = -\sum_{j=1}^{m} \bar{R}_{jijn} = -\sum_{j=1}^{m} (K_{ij;j} - K_{jj;i})$$

Emphasis: $G_{in} = -(\mathrm{div}_g(K) - d(\mathrm{tr}_g(K))_i = -\mathrm{div}_g(K - (\mathrm{tr}_g(K))g)_i$.

Now, by the Einstein equation, $G_{in} = \kappa T_{in} = -\kappa J_i$, which finishes the proof.

Discussion

There are lots of solutions to the Einstein Constraint Equations. In fact, if we don't restrict T in any way, then any space-like hypersurface in *any* Lorentzian manifold will satisfy the constraints.

If we impose restrictions on T, like the dominant energy condition, or the vacuum condition $T = 0$, then that restricts the geometry of $(\mathcal{S}, \bar{g})$ in some way, but still, of course, any Riemannian hypersurface yields a solution to the constraints.

Keep in mind the constraints form an underdetermined system, so in fact we might expect there to be a wide variety of solutions. In fact, from the point of view of the initial value formulation, this variety is useful for attempting to model a variety of physical situations.

Examples

Example

$(\mathcal{S}, \bar{g}) = (\mathbb{R}^{m+1}, \eta)$, Minkowski space-time, $\eta = -dt^2 + \sum_{i=1}^{m}(dx^i)^2$.

- $M = \{t = 0\}$. With the induced metric g, $(M, g) \cong (\mathbb{R}^m, g_{\mathrm{eucl}})$ and clearly $\overline{\nabla}_X n = 0$, so that $K = 0$. The constraints are trivial to verify.
- $M = \{-t^2 + \|x\|^2 = -1\}$. (M, g) is congruent to hyperbolic space $\mathbb{H}^m \cong (\mathbb{R}^m, g_{\mathbb{H}^m})$ of curvature -1. Moreover, since $n = x^\mu \frac{\partial}{\partial x^\mu}$ $(x^0 := t)$, for Y tangent to M, $\overline{\nabla}_Y n = Y$, so $K = -g$. One can easily verify the vacuum constraints.

Examples

If one imposes rotational symmetry on a solution to the Einstein vacuum equation $\mathrm{Ric}(\bar{g}) = 0$, one obtains the following solutions ($G = 1$, $c = 1$, $m = 3$), with coordinates (t, x^i), $\tilde{r} = |x|$, and $g_{\mathbb{S}^2} = d\Omega^2$ is the round unit sphere metric:

The Schwarzschild solution

$$\bar{g}_S = -\left(1 - \tfrac{2m}{\tilde{r}}\right) dt^2 + \left(1 - \tfrac{2m}{\tilde{r}}\right)^{-1} d\tilde{r}^2 + \tilde{r}^2 g_{\mathbb{S}^2}.$$

m is a constant of integration in one of the second-order ODE obtained; a second constant has been re-scaled to 1.

m is called the *mass*.

Note that the space-time is asymptotically Minkowksian as $\tilde{r} \to +\infty$.

Example (Schwarzschild)

$M = \{t = 0\}$ with induced metric $g_S = (1 - \frac{2m}{\tilde{r}})^{-1} d\tilde{r}^2 + \tilde{r}^2 g_{\mathbb{S}^2}$.

g_S must solve the vacuum Einstein constraint equation.

In fact, the second fundamental form vanishes:
$\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial t} \rangle = \langle \overline{\Gamma}^0_{ij} \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle = 0$. (Easy exercise)

So $K = 0$ (time-symmetric), and the vacuum constraint equations reduce to the vanishing of the scalar curvature: $R(g_S) = 0$.

Examples

We can re-write g_S in a convenient form: $g_S = (u(r))^4(dr^2 + r^2 g_{\mathbb{S}^2})$
Make a radial change of coordinate, using a positive increasing function r of $\tilde{r}$. We get

$g_S = (1 - \frac{2m}{\tilde{r}})^{-1} \left(\frac{d\tilde{r}}{dr}\right)^2 dr^2 + \left(\frac{\tilde{r}}{r}\right)^2 r^2 g_{\mathbb{S}^2}$.

We arrange

$$\left(1 - \frac{2m}{\tilde{r}}\right)^{-1/2} \frac{d\tilde{r}}{dr} = \frac{\tilde{r}}{r}$$

so that we can factor out of the metric g_S to obtain the proposed *conformally flat* representation of g_S.

We can solve the ODE (by separation, and the substitution $\tilde{r} = m + m\cosh w$, and imposing $\frac{\tilde{r}}{r} \to 1$ as $\tilde{r} \to +\infty$) to get $\frac{\tilde{r}}{r} = \left(1 + \frac{m}{2r}\right)^2$.

Examples

We thus obtain the following form of the Schwarzschild metric:

$$\bar{g}_S = -\frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^2} dt^2 + \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 g_{\mathbb{S}^2}\right)$$

$g_S = \left(1 + \frac{m}{2r}\right)^4 g_{\mathrm{Eucl}}$ is conformally flat, with $R(g_S) = 0$, which corresponds to the fact that $\left(1 + \frac{m}{2|x|}\right)$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$.

There is a higher-dimensional analogue too!

Example (Schwarzschild $m > 0$)

$(\mathbb{R}^n \setminus \{0\}, g_S, K = 0)$ with
$$g_S = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{4/(n-2)} g_{\mathrm{Eucl}}.$$

Question: What if $m < 0$? Compare the area profile $A(r) := \mathrm{Area}(\Sigma_r)$, where $\Sigma_r = \{|x| = r\}$ in the case $m > 0$ to the case $m < 0$.

Initial Value Formulation

Note that the constraints come from imposing $G_\Lambda(\bar{g})(n,\cdot) = \kappa T(n,\cdot)$, and as we saw above, this does not involve time derivatives of the space-time metric, but can be expressed in terms of g and K on the slice. This is qualitatively similar to Maxwell's equations, where the divergence equations must be satisfied at $t = 0$; this restricts the allowable initial data for the electromagnetic fields.

The Einstein constraint equations are necessary for (M, g, K) to be a space-like slice of a space-time $(\mathcal{S}, \bar{g})$ satisfying the Einstein equation. Question: Are the equations also sufficient, say in the vacuum case, or with suitable matter models? We remark that we are not imposing the full Gauss and Codazzi equations; in fact, we haven't imposed the space-time $(\mathcal{S}, \bar{g})$ into which we are embedding (M, g, K) either!

Answer: Yes! Y. Choquet-Bruhat, 1952. (M, g, K) can be interpreted as initial data for a Cauchy problem for Einstein's equation, which in suitable coordinates say, can be represented as a nonlinear hyperbolic system (cf. Hans Ringström's lectures).

Initial value formulation

With the initial-value problem in mind, we proceed to describe space-times split into a $(3+1)$ product structure.
Consider a Lorentzian manifold $(\mathcal{S} = I \times M, \bar{g})$, where $I \ni 0$ is an interval, and where the *slices* $M_t = \{t\} \times M$ are space-like. We let $g = g(t)$ be the induced metric on M_t. Let n be time-like unit normal to the slices, parallel to the space-time gradient of t, pointing in the same time direction as $\frac{\partial}{\partial t} = Nn + X$, where $\langle X, n \rangle \equiv 0$, and $N > 0$. N is the *lapse function*, and X is the *shift vector field*. We can write the metric in local coordinates x^i for M, with $X = X^i \frac{\partial}{\partial x^i}$,

$$\bar{g} = -N^2 \, dt^2 + g_{ij}(dx^i + X^i \, dt) \otimes (dx^j + X^j \, dt) \tag{1}$$

Note that $\bar{g}_{00} := \bar{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -(N^2 - |X|^2_g)$.
Also note that the Einstein summation convention here and in this section will be over the spatial indices.

Initial value formulation

If $(\mathcal{S}, \bar{g})$ satisfies a certain form of the Einstein equation, say vacuum for example, then the first and second fundamental forms $(g(t), K(t))$ of the slices M_t form a family of solutions to the Einstein constraint equations.

We can seek to turn this around a bit: what if we prescribed (suitable) N and X on $I \times M$ (e.g. $N = 1$, $X = 0$ would be a simple suitable choice), and seek to solve for $g = g(t)$ so that

$\bar{g} = -N^2\, dt^2 + g_{ij}(dx^i + X^i\, dt) \otimes (dx^j + X^j\, dt)$ satisfies the Einstein equation (in vacuum, say) on $\mathcal{S} \subset I \times M$, where $\mathcal{S} \supset \{0\} \times M$?

What equation would this impose on $g(t)$? $K(t)$?

Evolution of g and K

With our convention on K, we have
$\overline{\nabla}_X Y = \nabla_X Y - K(X,Y)n = \nabla_X Y + \hat{K}(X,Y)n$, where X and Y are tangent to a slice. We suppress the "t" subscript on M_t.
We will compute the time derivative of the induced metric and second fundamental form.
Recall that the Lie derivative of a tensor field is given by the product rule; for example, with connection D with $[X,Y] = D_X Y - D_Y X$,

$$\begin{aligned}(\mathcal{L}_X T)(Y,Z) &= X[T(Y,Z)] - T(\mathcal{L}_X Y, Z) - T(Y, \mathcal{L}_X Z)\\ &= X[T(Y,Z)] - T([X,Y],Z) - T(Y,[X,Z])\\ &= (D_X T)(Y,Z) + T(D_Y X, Z) + T(Y, D_Z X)\end{aligned}$$

For example, for $D = \nabla$ compatible with g ($\nabla g = 0$),
$(\mathcal{L}_X g)_{ij} = X_{i;j} + X_{j;i}$, where a semi-colon indicates covariant differentiation (whereas as a comma denotes a partial derivative):
$X_{i;j} = \nabla X(\partial_i, \partial_j) = (\nabla_{\partial_j} X)_i = \partial_j X_i - \Gamma^k_{ij} X_k$.

Evolution of g and K

Let $\partial_i = \frac{\partial}{\partial x^i}$ be a coordinate frame for M, and let $\partial_t = \frac{\partial}{\partial t}$. Using metric compatibility, the torsion-free property of the connection, and the fact that all the ∂_μ commute, we have

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= \overline{\nabla}_{\partial_t}\langle \partial_i, \partial_j\rangle \\
&= \langle \overline{\nabla}_{\partial_i}\partial_t, \partial_j\rangle + \langle \partial_i, \overline{\nabla}_{\partial_j}\partial_t\rangle \\
&= \langle \overline{\nabla}_{\partial_i}(Nn + X), \partial_j\rangle + \langle \partial_i, \overline{\nabla}_{\partial_j}(Nn + X)\rangle \\
&= N\langle \overline{\nabla}_{\partial_i} n, \partial_j\rangle + N\langle \partial_i, \overline{\nabla}_{\partial_j} n) + \langle \overline{\nabla}_{\partial_i} X, \partial_j\rangle + \langle \partial_i, \overline{\nabla}_{\partial_j} X\rangle \\
&= -2NK_{ij} + \langle \nabla_{\partial_i} X, \partial_j\rangle + \langle \partial_i, \nabla_{\partial_j} X\rangle \\
&= -2NK_{ij} + X^k_{;i} g_{kj} + X^k_{;j} g_{ik} \\
&= -2NK_{ij} + (\mathcal{L}_X g)_{ij} = 2N\hat{K}_{ij} + (\mathcal{L}_X g)_{ij}
\end{aligned}$$

where the semi-colon indicates covariant differentiation for the Levi-Civita connection ∇ of g.

Evolution of g and K

Evolution of g

$\frac{\partial g_{ij}}{\partial t} = 2N\hat{K}_{ij} + (\mathcal{L}_X g)_{ij}$

Note that the evolution of g is basically the definition of the second fundamental form, for which we can solve:
$K_{ij} = -\hat{K}_{ij} = -\frac{1}{2}N^{-1}\left(\frac{\partial g_{ij}}{\partial t} - (\mathcal{L}_X g)_{ij}\right)$.

Note: If we choose $N = 1$ and $X = 0$, we see $\partial_t g_{ij} = 2\hat{K}_{ij}$.

A more laborious exercise determines the time evolution of K.

Evolution of K

$\frac{\partial \hat{K}_{ij}}{\partial t} = N_{;ij} + (\mathcal{L}_X \hat{K})_{ij} + N(\bar{R}_{ij} - R_{ij} + 2\hat{K}_i^{\ell}\hat{K}_{j\ell} - \hat{K}^{\ell}_{\ell}\hat{K}_{ij})$
where $\bar{R}_{ij}$ are components of $\mathrm{Ric}(\bar{g})$, and R_{ij} are components of $\mathrm{Ric}(g)$.

We indicate how to start the proof of the evolution of $\hat{K}$. We recall $\frac{\partial}{\partial t} = Nn + X$.

Lemma

For $Y = Y^i \frac{\partial}{\partial x^i}$,

$$[n, Y] = (n[Y^i] + N^{-1} Y[X^i]) \frac{\partial}{\partial x^i} + N^{-1} Y[N] n.$$

In particular, then, $[n, \frac{\partial}{\partial x^\ell}]^{\mathrm{Tan}} = -N^{-1}[X, \frac{\partial}{\partial x^\ell}]$.

For future use, we also define $\hat{K}^2_{ij} = \hat{K}_{ik}\hat{K}_{\ell j} g^{k\ell} = \hat{K}^{\ell}_{i}\hat{K}_{\ell j}$ and $\mathrm{Hess}_g(N)_{ij} = N_{;ij}$, $\Delta_g N = \mathrm{tr}_g(\mathrm{Hess}_g(N)) = g^{ij} N_{;ij}$.

Evolution of g and K

Proof: We just start computing. We extend $\hat{K}$ to $T_p\mathcal{S}$ by $\hat{K}(v,w) = \hat{K}(v^{\mathrm{Tan}}, w^{\mathrm{Tan}})$, where the tangential projection is given by $v^{\mathrm{Tan}} = v + \langle v, n\rangle n$, and recall that $\overline{\nabla}_{\partial i} n$ is tangent to M; in fact $\overline{\nabla}_{\partial_i} n = \hat{K}_{ik} g^{k\ell}\partial_\ell$:

$$
\begin{aligned}
\partial_t \hat{K}_{ij} &= \overline{\nabla}_{Nn+X}[\hat{K}(\partial_i,\partial_j)] \\
&= N(\overline{\nabla}_n \hat{K})_{ij} + N\hat{K}(\overline{\nabla}_n \partial_i, \partial_j) + N\hat{K}(\partial_i, \overline{\nabla}_n \partial_j) + X[\hat{K}_{ij}] \\
&= N(\overline{\nabla}_n \hat{K})_{ij} + N\hat{K}(\overline{\nabla}_{\partial_i} n + [n,\partial_i], \partial_j) + N\hat{K}(\partial_i, \overline{\nabla}_{\partial_j} n + [n,\partial_j]) + X[\hat{K}_{ij}] \\
&= N(\overline{\nabla}_n \hat{K})_{ij} + N\hat{K}(\overline{\nabla}_{\partial_i} n, \partial_j) + N\hat{K}(\partial_i, \overline{\nabla}_{\partial_j} n) \\
&\quad + N\hat{K}(-N^{-1}[X,\partial_i], \partial_j) + N\hat{K}(\partial_i, -N^{-1}[X,\partial_j]) + X[\hat{K}_{ij}] \\
&= N(\overline{\nabla}_n \hat{K})_{ij} + N\hat{K}(\hat{K}_{ik} g^{k\ell}\partial_\ell, \partial_j) + N\hat{K}(\partial_i, \hat{K}_{jk} g^{k\ell}\partial_\ell) + (\mathcal{L}_X \hat{K})_{ij} \\
&= N(\overline{\nabla}_n \hat{K})_{ij} + 2N\hat{K}_{ik}\hat{K}_{\ell j} g^{k\ell} + (\mathcal{L}_X \hat{K})_{ij}
\end{aligned}
$$

where in the last step we use the symmetry of $\hat{K}$.

Evolution of g and K

It remains to relate $\overline{\nabla}_n \hat{K}$ to the curvature. Such a relation is given by a Mainardi equation, computing an (n, n) component of the curvature, applied to $\bar{g} = -N^2 dt^2 + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt)$.

Mainardi equation

For $\langle Y, n\rangle \equiv 0 \equiv \langle Z, n\rangle$,

$$\langle \bar{R}(Y, n, n), Z\rangle = -(\overline{\nabla}_n \hat{K})(Y, Z) - \hat{K}^2(Y, Z) + N^{-1}\mathrm{Hess}_g(N)(Y, Z)$$

The proof is a great exercise to see if you've digested all the various relationships! To start you off:
$(\overline{\nabla}_n \hat{K})(Y, Z) = \overline{\nabla}_n(\hat{K}(Y, Z)) - \hat{K}(\overline{\nabla}_n Y, Z) - \hat{K}(Y, \overline{\nabla}_n Z)$. Write $\hat{K}(Y, Z) = \langle \overline{\nabla}_Y n, Z\rangle$ in the first term, and in the second and third term, switch the order of covariant differentiation, which brings in commutator terms. Remember to take tangential components inside $\hat{K}$. Turn the crank and enjoy the ride.

Evolution of g and K

From here and above, we conclude the evolution of $\hat{K}$ with the following elementary lemma, which is proved in a manner similar to the Hamiltonian constraint, using the Gauss equation (Exercise!):

Lemma

$$\langle \bar{R}(Y,n,n),Z\rangle = -\overline{\mathrm{Ric}}(Y,Z) + \mathrm{Ric}(Y,Z) + (\mathrm{tr}_g \hat{K})\hat{K}(Y,Z) - \hat{K}^2(Y,Z).$$

To summarize, in index form: from the above and Mainardi

$$\langle \bar{R}(Y,n,n),Z\rangle = -(\overline{\nabla}_n \hat{K})(Y,Z) - \hat{K}^2(Y,Z) + N^{-1}\mathrm{Hess}_g(N)(Y,Z)$$

we obtain

$$(\overline{\nabla}_n \hat{K})_{ij} = \bar{R}_{ij} - R_{ij} - (\mathrm{tr}_g \hat{K})\hat{K}_{ij} + N^{-1}N_{;ij}.$$

Evolution of g and K

We have thus arrived at the following system:

ADM equations

$$\frac{\partial g_{ij}}{\partial t} = 2N\hat{K}_{ij} + (\mathcal{L}_X g)_{ij}$$
$$\frac{\partial \hat{K}_{ij}}{\partial t} = N_{;ij} + (\mathcal{L}_X \hat{K})_{ij} + N(\bar{R}_{ij} - R_{ij} + 2\hat{K}_i^{\ell}\hat{K}_{j\ell} - \hat{K}_{\ell}^{\ell}\hat{K}_{ij})$$

The space-time Ricci term $\bar{R}_{ij}$ in the evolution equation above can be expressed in terms of the Einstein tensor: for $\dim(\mathcal{S}) = 4$, $\bar{R}_{\mu\nu} = G_{\mu\nu} - \frac{1}{2}(\mathrm{tr}_{\bar{g}} G)\bar{g}_{\mu\nu}$.
In particular, in the vacuum case ($T = 0$, and take $\Lambda = 0$), $\bar{R}_{ij} = 0$ gives six components of the vacuum Einstein equation $G_{\mu\nu} = 0$ ($\bar{R}_{\mu\nu} = 0$). The other four components are given by the *Einstein constraint equations*, which we saw encode G_{nn} and G_{in}.

Evolution of g and K

If we specify the lapse $N > 0$ and shift X, and with an eye toward the vacuum equations we insert $\bar{R}_{ij} = 0$ into the system, if we can solve the system for $g = g(t)$, the first equation guarantees that $\hat{K} = -K$ is the second fundamental form of M_t inside the space-time $(\mathcal{S}, \bar{g} = -N^2 dt^2 + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt))$.

In so doing, we have $\bar{R}_{ij} = 0$. As for the other components of the vacuum Einstein equation, we need to specify initial conditions $(g(0), K(0)$. If we specify these so that they above the vacuum constraint equations, then $G_{\mu n} = 0$ at $t = 0$.

What we would like, then, is as follows:

- PDE framework allowing to solve the system and estimate quantities related to the solution.
- Evolution equations for the constraint functions (which encode $G_{\mu n}$), from which we can conclude the constraint equations are conserved in time.

Conservation of the constraints

We could compute the time evolution of the constraint functions $R(g) - \|K^2\|_g + (\mathrm{tr}_g K)^2$ and $\mathrm{div}_g(K - (\mathrm{tr}_g K))$, and show that they satisfy a system of PDE, of a form in which should they vanish at $t = 0$ (the vacuum constraints imposed), then they vanish for all t.

In any case, the conservation of the constraints comes from the *Bianchi identity*: $\bar{g}^{\nu\beta} G_{\mu\nu;\beta} = 0$. For convenience to illustrate, take $N = 1$, $X = 0$, so that $n = \frac{\partial}{\partial t}$, so that $\bar{g}^{00} = -1 = \bar{g}_{00}$, and so we get (sign!)

$$\partial_t G_{\mu 0} - \bar{\Gamma}^{\beta}_{\mu 0} G_{\beta 0} - \bar{\Gamma}^{\beta}_{00} G_{\mu\beta} = \bar{g}^{\nu k} G_{\mu\nu;k} = \bar{g}^{jk} G_{\mu j;k}.$$

Since we arrange $G_{ij} = 0$ for all t through solving the system, we can remove G_{ij} and $G_{ij,k}$ from the above equation. So, if I did this right,

$$\partial_t G_{i0} = \bar{\Gamma}^{\beta}_{i0} G_{\beta 0} - \bar{g}^{jk} \bar{\Gamma}^{0}_{ik} G_{0j} - \bar{g}^{jk} \bar{\Gamma}^{0}_{jk} G_{0i}$$
$$\partial_t G_{00} = \bar{g}^{ij} \partial_i G_{0j} + \bar{\Gamma}^{\beta}_{00} G_{\beta 0} + \bar{\Gamma}^{i}_{00} G_{0i} - \bar{g}^{ij} \bar{\Gamma}^{0}_{0i} G_{0j} - \bar{g}^{ij} \bar{\Gamma}^{\beta}_{ij} G_{0\beta}$$

Linear and homogeneous for $G_{\mu 0}$, which vanish at $t = 0$ by the constraints.

Hamiltonian formulation

In classical mechanics, one often re-casts the Lagrangian L in the form $L = \int_{t_1}^{t_2} (p\dot{q} - H(p,q))\, dt$, where q are the state variables and p are the conjugate momenta. Stationarity of the action L then gives Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}.$$

We can recast the Einstein-Hilbert action $\int_{\mathcal{S}} \bar{R}(\bar{g})\, d\mu_{\bar{g}}$ on $(\mathcal{S} = I \times M, \bar{g})$ in an analogous fashion. Indeed, we want to consider stationary points of the EH action, so when we re-write the action, we will discard boundary terms which we can take to vanish when deriving the Euler-Lagrange equations. The result can be written as follows.

Hamiltonian formulation

We consider $\mathcal{S} = I \times M$, $\bar{g} = -N^2 dt^2 + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt)$. We express the scalar curvature $R(\bar{g})$ using the Gauss and Mainardi equations, as follows, using an orthonormal frame $\{n, e_i\}$ adapted to M.

$$\begin{aligned}
R(\bar{g}) &= \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} = -\bar{R}_{nn} + \sum_j \bar{R}_{jj} \\
&= -\bar{R}_{nn} + \sum_j \Big[-\langle \bar{R}(n, e_j, e_j), n\rangle + \sum_i \langle \bar{R}(e_i, e_j, e_j), e_i\rangle \Big] \\
&= -2\sum_j \bar{R}_{jnnj} + \sum_{i,j} \bar{R}_{ijji} \\
&= -2\sum_j \Big[-(\overline{\nabla}_n \hat{K})_{jj} - K_j{}^{\ell} K_{\ell j} + N^{-1} N_{;jj} \Big] + \sum_{i,j} \Big[R_{ijji} + K_{ii}K_{jj} - K_{ij}^2 \Big] \\
&= 2\overline{\nabla}_n(\mathrm{tr}_g \hat{K}) - 2N^{-1}\Delta_g N + 2\|K\|_g^2 + (R(g) - \|K\|_g^2 + (\mathrm{tr}_g K)^2)
\end{aligned}$$

We note that $\overline{\nabla}_n(\mathrm{tr}_g(\hat{K})) = N^{-1}\Big(\partial_t(\mathrm{tr}_g(\hat{K})) - \nabla_X(\mathrm{tr}_g \hat{K})\Big)$.

Hamiltonian formulation

Let the constraints operator $\Phi = (\Phi_0, \Phi_i)$ be given by
$\Phi(g, K) = (R(g) - \|K\|_g^2 + (\mathrm{tr}_g K)^2, \mathrm{div}_g(K - (\mathrm{tr}_g)K))$, let
$\widehat{\Phi} = (\Phi_0(g, K), -2\Phi_i(g, K))$. Let $\xi^0 = N$, $\xi^i = X^i$.
Then, modulo boundary terms, $\int\limits_{\mathcal{S}=I\times M} \bar{R}(\bar{g})\, d\mu_{\bar{g}} = \int\limits_I \int\limits_M R(\bar{g}) N d\mu_g dt$, is given by (Exercise! Integrate by parts, use $\frac{d}{dt}\log\det(g) = \mathrm{tr}_g(\partial_t g)$...)

$$\int\limits_I \int\limits_M \left[(\hat{K}^{ij} - (\mathrm{tr}_g \hat{K}) g^{ij})\partial_t g_{ij} + \xi^\mu \widehat{\Phi}_\mu\right] d\mu_g\, dt.$$

This is then re-written as $\int\limits_I \left[\int\limits_M \hat{\pi}^{ij}\partial_t g_{ij}\, d\mu_g - H_{ADM}\right] dt$. Note that often the conjugate momentum $\hat{\pi}$ is written as a density,
$\tilde{\pi}^{ij} = \sqrt{\det g}(\hat{K}^{ij} - (\mathrm{tr}_g \hat{K}) g^{ij})$.
Note that the lapse $\xi^0 = N$ and shift $\xi^i = X^i$ appear as Lagrange multipliers for a constrained optimization problem: stationarity of the action with respect to variations of lapse give the Hamiltonian constraint $\Phi_0 = 0$, and similarly for the shift.

Hamiltonian formulation

Let $\mathcal{H}_{\rm tot} = -\xi^\mu \widehat{\Phi}_\mu$, so that $H_{ADM} = \int\limits_M \mathcal{H}_{\rm tot}$. Varying the action by $\hat{\pi}^{ij} \mapsto \hat{\pi}^{ij} + \epsilon\hat{\sigma}^{ij}$ and applying $\frac{d}{d\epsilon}\Big|_{\epsilon=0}$ we obtain by definition $\int\limits_I \int\limits_M \hat{\sigma}^{ij}\left[\partial_t g_{ij} - \frac{\partial\mathcal{H}_{\rm tot}}{\partial\pi^{ij}}\right] d\mu_g \, dt$.

We formally arrive at one of Hamilton's equations $\frac{\partial g_{ij}}{\partial t} = \frac{\partial\mathcal{H}_{\rm tot}}{\partial\pi^{ij}}$. To figure out what this means, we must compute

$$\int\limits_M \frac{\partial\mathcal{H}_{\rm tot}}{\partial\pi^{ij}}\hat{\sigma}^{ij} \, d\mu_g := \int\limits_M \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}_{\rm tot} d\mu_g.$$

An easy exercise (integrate by parts!) shows this is just

$$\int\limits_M \hat{\sigma}^{ij}\left(2N\hat{K}_{ij} + X_{i;j} + X_{j;i}\right) d\mu_g.$$

We thus recover the first of the ADM equations $\partial_t g_{ij} = 2N\hat{K}_{ij} + (\mathcal{L}_X g)_{ij}$.

Evolution and the linearization of the constraints map

We let $\widetilde{\Phi}(g,\tilde{\pi}) = \sqrt{\det g}(\Phi_0, 2\Phi_i)$. Then

$$\frac{d}{dt}\begin{bmatrix} g \\ \tilde{\pi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} D\tilde{\Phi}|^*_{(g,\tilde{\pi})}(N,X)$$

where if we let $d\tilde{\mu}_g = (\det g)^{-1/2} d\mu_g$, then (assuming I sorted out all the square roots)

$$\int_M (N,X) \cdot_g D\widetilde{\Phi}|_{(g,\tilde{\pi})}(h,\tilde{\sigma}) d\tilde{\mu}_g = \int_M D\tilde{\Phi}|^*_{(g,\tilde{\pi})}(N,X) \cdot_g (h,\tilde{\sigma}) d\tilde{\mu}_g.$$

In this sense, the constraints determine the evolution. Furthermore, the symplectic form of the system above points us to consider the kernel of the operator $D\Phi^*$ (defined in a similar way without the densitized fields and measures, etc.).

The operator admits kernel only in certain special situations, and this kernel is important for understanding the properties of the nonlinear constraints map, in terms of deformations (implicit function theorem).

Constraints operator and its linearization

We write $\Phi(g,K) = (R(g) - \|K\|_g^2 + (\mathrm{tr}_g K)^2, \mathrm{div}_g(K - (\mathrm{tr}_g K)g))$. If we use the momentum tensor $\pi^{ij} = K^{ij} - (\mathrm{tr}_g K)g^{ij}$ instead, the constraints operator can be replaced by $\Phi(g,\pi) = (R(g) - \|\pi|_g^2 + \frac{1}{2}(\mathrm{tr}_g \pi)^2, \mathrm{div}_g \pi)$, where the output is now a function and a vector field.

We can linearize with respect to variations in g and π, and then define the formal adjoint $D\Phi^*$ analogously to the above (remove the tilde's):

$$\int_M (N,X) \cdot_g D\Phi|_{(g,\pi)}(h,\sigma) d\mu_g = \int_M D\Phi|^*_{(g,\pi)}(N,X) \cdot_g (h,\sigma) d\mu_g.$$

The general formula is a bit cumbersome, but let's look at a special case:

$$D\Phi^*|_{(g_{\mathrm{Eucl}},0)}(N,X) = (L^*_{g_{\mathrm{Eucl}}} N, -\frac{1}{2}(\mathcal{L}_X g_{\mathrm{Eucl}})^\sharp)$$

where L_g^* is the formal adjoint of L_g, $L_g(h) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} R(g+\epsilon h)$.

Constraints operator and its linearization

Exercise: $L_g(h) = -\Delta_g(\mathrm{tr}_g h) + \mathrm{div}_g(\mathrm{div}_g h) - h^{ij}R_{ij}$.

Thus by integration by parts, $L_g^* N = -(\Delta_g N)g + \mathrm{Hess}_g N - N\mathrm{Ric}(g)$.

Note that $\mathrm{tr}_g(L_g^* N) = -(n-1)\Delta_g N - NR(g)$.

Example (Kernel of L_g^*)

- Euclidean space $\mathbb{R}^n$, with $L_{g_{\mathrm{E}}}^* N = 0$ if and only if $\mathrm{Hess}_{g_{\mathrm{E}}} N = 0$, so the kernel is spanned by constant and linear functions of Cartesian coordinates.
- The flat torus $\mathbb{T}^n$ with one-dimensional kernel given by constant functions.
- The round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, with basis for the kernel given by restriction of the coordinate functions $x^j\big|_{\mathbb{S}^n}$, $j = 1, \ldots, n+1$.

Constraints operator and its linearization

Definition

A nontrivial element in the kernel of L_g^* is called a static potential

If $L_g^* N = 0$, then $\bar{g} = -N^2 dt^2 + g$ is Einstein: $\mathrm{Ric}(\bar{g}) = \frac{R(g)}{n-1}\bar{g}$, where $n = \dim(M)$ (note the change—"m" will be the mass anyway...). In particular, we can conclude that $R(g)$ is (locally) constant.

Example (Schwarzschild)

$$\bar{g}_S = -\frac{\left(1 - \frac{m}{2|x|}\right)^2}{\left(1 + \frac{m}{2|x|}\right)^2}\, dt^2 + \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathrm{Eucl}}$$

$N = \frac{\left(1-\frac{m}{2|x|}\right)}{\left(1+\frac{m}{2|x|}\right)}$ is in the kernel of $L_{g_S}^*$.

Remark: what is special about $\{N = 0\}$ in terms of the geometry of g_S? (Hint: area profile $A(r) = A(\{|x| = r\})$.)

Constraints operator and its linearization

The scalar curvature operator is the constraints operator in the time-symmetric case $K = 0$.

As for the full constraint operator, we note an example:

Example (Minkowski)

$D\Phi^*|_{(g_{\mathrm{Eucl}},0)}(N, X) = (L^*_{g_{\mathrm{Eucl}}} N, -\frac{1}{2}(\mathcal{L}_X g_{\mathrm{Eucl}})^\sharp)$

Thus the kernel is the direct sum of the span of linear and constant functions, together with the Killing fields of the Euclidean metric, spanned by generators of translations and rotations.

The presence of kernel again marks something special about (g, π). Note above how the kernel elements generate Killing vector fields in the space-time: $N\frac{\partial}{\partial t} + X$. Recalling the formula

$$\frac{d}{dt}\begin{bmatrix} g \\ \tilde{\pi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} D\tilde{\Phi}|^*_{(g,\tilde{\pi})}(N, X)$$

the following theorem may not be a surprise:

Constraints operator and its linearization

Theorem (Moncrief, '75)

Suppose (M, g, π) solve the vacuum constraints. If (N, X) is in the kernel of $D\Phi^*_{(g,\pi)}$, the vacuum space-time determined by (M, g, π) admits a Killing field, whose normal and tangential projections along M yield N and X, respectively.

Such space-times are called stationary (if the Killing field is time-like). As such, (N, X) in the kernel is called a KID—Killing Initial Data. In the time-symmetric case, an element in the kernel of L_g^* is called a static KID.

Asymptotically Euclidean metrics

We now discuss a class of solutions to the constraints, often used to model isolated systems.

Definition

A complete Riemannian manifold (M^n, g), with an associated symmetric $(0,2)$ tensor K (or $\pi = K - (\mathrm{tr}_g K)g$) is called *asymptotically flat* (or *asymptotically Euclidean*) if there is a compact set $\mathcal{K} \subset M$ so that $M \setminus \mathcal{K}$ equals a disjoint union of asymptotic ends $\bigcup_{j=1}^{k} E_j$, where each asymptotic end E_j is diffeomorphic to $\mathbb{R}^n \setminus \{|x| \leq 1\}$, with asymptotically flat coordinates x^i in which the following decay estimates hold for multi-indices α and β, for $|\alpha| \leq \ell + 1$, $|\beta| \leq \ell$, $q > \frac{n-2}{2}$

$$\begin{aligned} |\partial_x^\alpha (g_{ij} - \delta_{ij})(x)| &= O(|x|^{-|\alpha|-q}) \\ \left|\partial_x^\beta K_{ij}(x)\right| &= O(|x|^{-|\beta|-1-q}) \end{aligned}$$

Asymptotically Euclidean metrics

Let's parse this in case $n = 3$, $q = 1 > \frac{3-2}{2}$:

$$\begin{aligned} |\partial_x^\alpha (g_{ij} - \delta_{ij})(x)| &= O(|x|^{-|\alpha|-1}) \\ \left|\partial_x^\beta K_{ij}(x)\right| &= O(|x|^{-|\beta|-2}) \end{aligned}$$

So, $g_{ij}(x) = \delta_{ij} + O(|x|^{-1})$, $|\partial_x g_{ij}(x)| = O(|x|^{-2})$, $\left|\partial_x^2 g_{ij}(x)\right| = O(|x|^{-3})$ etc., $K_{ij}(x) = O(|x|^{-2})$, $|\partial_x K_{ij}(x)| = O(|x|^{-3})$, etc.

Example (Schwarzschild)

$(g_S)_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} = (1 + \frac{2m}{|x|} + \frac{3m^2}{2|x|^2} + \frac{m^3}{2|x|^3} + \frac{m^4}{16|x|^4})\delta_{ij}$
$K = 0$

In dimensions $n \geq 3$, $g_S = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{4/(n-2)} g_{\mathrm{E}}$ gives examples with $q = n - 2$.

Energy-momentum four vector

• We'd like to define a quantity to measure the energy-momentum content of an isolated gravitational system. In terms of the data on a space-like slice, we do know that the energy density is related to the scalar curvature.
• Note that the vacuum constraint equations impose added decay to the scalar curvature than just the asymptotic conditions alone: if $n = 3$ and $q = 1$, say, $R(g)$ involves two derivatives of the metric (and quadratic terms in Christoffel symbols), so that $R(g) = O(|x|^{-3})$ on any end E.
• If the vacuum constraints are satisfied, then $R(g)$ is quadratic in $K = O(|x|^{-2})$, so that $R(g) = O(|x|^{-4})$. This means that $R(g) \in L^1(E)$, and similarly if ρ is integrable: $R(g) - \|K\|_g^2 + (\mathrm{tr}_g K)^2 = 2\kappa\rho$.
•In case $K = 0$, then, one might try to define the total energy as proportional to $\int_M R(g) d\mu_g$.
• This doesn't pick up the energy due to the gravitational field, which can have energy in vacuum—cf. Schwarzschild g_S for example.
This means the definition must be a bit more subtle.

Newtonian analogue

In Newtonian theory, we can measure the total mass in an isolated system through a flux integral of the gravitational potential: if ϕ is the gravitational potential, then ϕ solves Poisson's equation $\Delta\phi = 4\pi\rho$ $(G = 1)$.

If the matter density ρ is compactly supported, ϕ is harmonic near infinity, and we can take $\phi(x) \to 0$ as $|x| \to \infty$

We can expand ϕ in spherical harmonics near infinity:

$$\phi(x) = -\frac{m}{|x|} - \frac{\beta \cdot x}{|x|^3} + O(|x|^{-3}).$$

Then we compute

$$\int\limits_{\mathbb{R}^3} \rho \, dx = \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \Delta\phi \, dx = \frac{1}{4\pi} \lim_{r\to+\infty} \int\limits_{\{|x|=r\}} \frac{\partial\phi}{\partial r} d\sigma_e = m.$$

Exercise: $\int\limits_{\mathbb{R}^3} x^k \rho \, dx = \beta^k$. The center of mass c^k would then be defined by $mc^k = \beta^k$.

Expansion of scalar curvature

In GR, the potential is replaced by the metric, so maybe appropriate flux integrals should be used to pick up the total mass/energy content of the system.

Let's just start with an asymptotically flat metric g, without imposing the constraints yet. We have $g_{ij} - \delta_{ij} = O(|x|^{-q})$, so that $g^{ij} - \delta^{ij} = O(|x|^{-q})$, $\partial g_{ij} = O(|x|^{-q-1})$ and $\partial^2 g_{ij} = O(|x|^{-q-2})$.

Now $\Gamma^k_{ij} = \frac{1}{2} g^{km}(g_{im,j} + g_{mj,i} - g_{ij,m}) = O(|x|^{-q-1})$.

$$\begin{aligned}
\Gamma^k_{ij,\ell} &= \frac{1}{2} g^{km}_{,\ell}(g_{im,j} + g_{mj,i} - g_{ij,m}) + \frac{1}{2} g^{km}(g_{im,j\ell} + g_{mj,i\ell} - g_{ij,m\ell}) \\
&= O(|x|^{-2q-2}) + \frac{1}{2}\delta^{km}(g_{im,j\ell} + g_{mj,i\ell} - g_{ij,m\ell}) \\
&+ \frac{1}{2}(g^{km} - \delta^{km})(g_{im,j\ell} + g_{mj,i\ell} - g_{ij,m\ell}) \\
&= \frac{1}{2}(g_{ik,j\ell} + g_{kj,i\ell} - g_{ij,k\ell}) + O(|x|^{-2q-2}).
\end{aligned}$$

Expansion of scalar curvature

Recall $R(g) = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{k\ell}\Gamma^\ell_{ij} - \Gamma^k_{j\ell}\Gamma^\ell_{ik}\right)$. Thus
$R(g) = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j}\right) + O(|x|^{-2q-2}) = \delta^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j}\right) + O(|x|^{-2q-2})$
because $g - \delta \sim |x|^{-q}$ and $\partial\Gamma \sim |x|^{-q-2}$.

Proposition

Thus using previous expansions,

$$R(g) = \sum_{i,j}(g_{ij,ij} - g_{ii,jj}) + O(|x|^{-2q-2}).$$

Note that the error term is in $L^1(E)$: for $q > (n-2)/2$, $2q+2 > n$.

Expansion of scalar curvature

If we write $g_{ij} = \delta_{ij} + h_{ij}$, then

$$\begin{aligned} R(g) &= \sum_{i,j}(h_{ij,ij} - h_{ii,jj}) + O(|x|^{-2q-2}) \\ &= -\sum_j \Big(\sum_i h_{ii} \Big)_{,jj} + \sum_j \Big(\sum_i h_{ij,i} \Big)_{,j} + O(|x|^{-2q-2}) \\ &= -\Delta_{g_{\mathrm{E}}}(\mathrm{tr}_{g_{\mathrm{E}}} h) + \mathrm{div}_{g_{\mathrm{E}}}(\mathrm{div}_{g_{\mathrm{E}}} h) + O(|x|^{-2q-2}) \\ &= L_{g_{\mathrm{E}}}(h) + O(|x|^{-2q-2}). \end{aligned}$$

Of course!

Note that $L^*_{g_{\mathrm{E}}}(1) = 0$, so the integral below is a flux integral:

$$\int_{\{r_1 \le |x| \le r_2\}} L_{g_{\mathrm{E}}}(h)\, dx = \Big[\int_{\{|x|=r_2\}} - \int_{\{|x|=r_1\}} \Big] \sum_{i,j}(h_{ij,i} - h_{ii,j})\, \nu_e^j \, d\sigma_e.$$

The mass (energy) integral

Energy

The mass (energy) integral is given by

$$\frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to+\infty} \int_{|x|=r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu_e^j \, d\sigma_e.$$

Note $\nu_e^j = \frac{x^j}{|x|}$.

In case $n=3$, $\omega_2 = 4\pi$, and this is $\frac{1}{16\pi} \lim_{r\to+\infty} \int_{|x|=r} \sum_{i,j=1}^{3} (g_{ij,i} - g_{ii,j})\nu_e^j \, d\sigma_e$.

Does the limit exist?

Exercise: For $g_S = \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathrm{E}}$, show this limit is precisely m.

The mass (energy) integral

More generally, by the divergence theorem,

$$\left[\int_{|x|=r} - \int_{|x|=r_0}\right] \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu_e^j \, d\sigma_e = \int_{r_0 \leq |x| \leq r} \sum_{i,j=1}^{n} (g_{ij,ij} - g_{ii,jj}) \, dx$$

$$= \int_{r_0 \leq |x| \leq r} \sum_{i,j=1}^{n} (R(g) + O(|x|^{-2q-2}) \, dx.$$

Constraints

So if $R(g) \in L^1$, then the limit exists.

From the constraint $R(g) - \|K\|_g^2 + (\mathrm{tr}_g(K))^2 = 2\kappa\rho$, with $K_{ij} = O(|x|^{-q-1})$, we see $R(g) \in L^1(E)$ if and only if $\rho \in L^1(E)$, e.g. vacuum case $\rho = 0$.

The energy-momentum four-vector

Remarks

- The limit can be non-zero even if $R(g) = 0$ (vacuum), e.g. Schwarzschild.
- One can show (Bartnik, CPAM, 1986) that the mass/energy is independent of AF coordinates.

Now recall the second constraint equation $\mathrm{div}_g(\pi) = \kappa J$. Now

$$(\mathrm{div}_g \pi)_i = g^{jk}(\pi_{ij,k} - \Gamma^m_{ik}\pi_{mj} - \Gamma^m_{kj}\pi_{im}) = \mathrm{div}_{g_{\mathrm{E}}}(\pi) + O(|x|^{-2q-2})$$

So for $|J| \in L^1(E)$, e.g. $J = 0$, the following limit exists:

$$P_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to+\infty} \int\limits_{|x|=r} \pi_{ij}\nu_e^j \, d\sigma_e.$$

Remark: $D\Phi^*_{(g_{\mathrm{E}},0)}(1, \frac{\partial}{\partial x^i}) = (0,0)$, $D\Phi_{(g_{\mathrm{E}},0)}(h,\pi) = (L_{g_{\mathrm{E}}}(h), \mathrm{div}_{g_{\mathrm{E}}}(\pi))$

The energy-momentum four-vector

If we let $E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to+\infty} \int_{|x|=r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu_e^j \, d\sigma_e$, then E and P fit together to given the *ADM energy-momentum four-vector* of the asymptotically flat end.

What do we expect?

For reasonable matter, say corresponding to T satisfying an energy condition, we expect $E \geq 0$, and in $E \geq |P|$, i.e. the energy-momentum vector is future-pointing causal.

This is the content of the *Positive Mass Theorem*.

Positive Mass Theorem

Recall that the dominant energy condition (DEC) implies on the initial data that $\rho \geq |J|$, i.e.

$$\frac{1}{2}(R(g) - \|K\|_g^2 + (\mathrm{tr}_g(K))^2) \geq |\mathrm{div}_g \pi|.$$

PMT

(M, g, K) is an asymptotically flat initial data set satisfying DEC. Then $E \geq |P|$.

PET

(M, g, K) is an asymptotically flat initial data set satisfying DEC. Then $E \geq 0$.

Positive Mass Theorem

Riemannian case

(M, g) is an asymptotically flat initial data set satisfying $R(g) \geq 0$. Then $E \geq 0$. $E = 0$ implies (M, g) isometric to $(\mathbb{R}^n, g_E)$.

Notes

- Riemannian case: Schoen-Yau, 1979 ($3 \leq n \leq 7$), Witten 1981 (spin manifolds).
- PET: Schoen-Yau, 1981 ($n = 3$), Witten (spin); Eichmair ($4 \leq n \leq 7$).
- PMT: Witten, 1981 (spin); Eichmair, Huang, Lee, Schoen, 2011 ($3 \leq n \leq 7$).

The rigidity statement in the PMT is: $E = |P|$ only in case M^n is a space-like hypersurface in Minkowski space-time $\mathbb{M}^{n+1}$ with induced metric and second fundamental form g and K. This is established in the spin case. Also, in case $E = 0$: Schoen-Yau, Eichmair.

Positive Mass Theorem

Please see arXiv preprint of Eichmair, Huang, Lee and Schoen (2011), and Eichmair (2012) for references and history.

If time permits we will sketch a proof the Riemannian case of the PET, at least in case $n = 3$. This involves a careful understanding of the scalar curvature and topology.

Remark: There is negative mass Schwarzschild g_S with $m < 0$, which does satisfy $R(g_S) = 0$. What's wrong with it?

There is a study of negative mass singularities initiated by Hugh Bray and students Nick Robbins, Jeff Jauregui.

Solving the constraints

- We'd like to have ways to generate solutions to the vacuum constraints (apart from taking slices in known space-times).
- We remark that the constraint system is undetermined, and we have lots of freedom. But the system is nonlinear and is non-trivial.
- Note, too, that the rigidity case of the PET tells us we cannot ask for AE solutions to fall off too fast—e.g. in dimension $n = 3$, we cannot ask the solution to satisfy $g_{ij} - \delta_{ij} = O(|x|^{-2})$, say, since then $E = 0$, and by the PET, the metric is isometric to Euclidean. The system dictates something about the asymptotics, cf. the Newtonian analogue/Poisson equation.
- Remark: Once we've solved the vacuum constraints with AE data, a natural question is: to what extent is the space-time development asymptotically Minkowskian? How "big" of a neighborhood of spatial infinity does the space-time include? Cf. Boost Theorem of Christodoulou and Ó Murchadha, Nonlinear Stability of Minkowski Space by Christodoulou and Klainerman, etc.

Solving the constraints

Let's focus on the time-symmetric vacuum constraint $R(g) = 0$ on $\mathbb{R}^3$ for the moment.

If h were compactly supported, then $g = g_{\mathrm{E}} + h$ can't be a nontrivial solution by the PMT.

Note, however, that the linearized equation $L_{g_{\mathrm{E}}}(h) = -\Delta_{g_{\mathrm{E}}}(\mathrm{tr}_{g_{\mathrm{E}}} h) + \mathrm{div}_{g_{\mathrm{E}}}(\mathrm{div}_{g_{\mathrm{E}}} h)$ does admit nontrivial compactly supported solutions!

Exercise: Show that there are nontrivial compactly supported symmetric tensors h on $\mathbb{R}^3$ which are divergence-free and trace-free (with respect to g_{E}). Such tensors are called transverse-traceless (TT).

Remark: The method we'll discuss shortly also gives a way to construct TT tensors which decay suitably at infinity to use in this method to construct AE solutions.

Now what? Well, we look for a conformal factor $u > 0$ to try to help us solve $R(g) = 0$ with $g = u^4(g_{\mathrm{E}} + h)$. Since we want the solution to be AE, we want $u \to 1$ at infinity.

Solving the constraints

For Riemannian manifolds (M, g) of dimension $n \geq 3$, we have the following formula for the scalar curvature under a conformal change (Exercise):

$$R(u^{4/(n-2)}g) = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}}(\Delta_g u - \frac{n-2}{4(n-1)} R(g)u).$$

Case $n = 3$: $R(u^4 g) = -8u^{-5}(\Delta_g u - \frac{1}{8} R(g)u)$

If $g = g_{\mathrm{E}}$ is Euclidean near infinity say, then prescribing $R(u^{\frac{4}{n-2}} g_{\mathrm{E}}) = 0$ is equivalent to $\Delta u = 0$, $R(u^{\frac{4}{n-2}} g_{\mathrm{E}}) \geq 0$ is equivalent to $\Delta u \leq 0$ (u is *superharmonic*). Why did I say $g = g_{\mathrm{E}}$ near infinity?.

More generally, prescribing the scalar curvature of the conformal metric gives a semi-linear elliptic equation; prescribing that it vanish gives the linear equation $\Delta_g u - \frac{n-2}{4(n-1)} R(g)u = 0$.

So it is important to understand the behavior of operators of the form $(\Delta_g - f)$.

Solving the constraints

• If we consider $g_h = g_{\mathrm{E}} + h$, $R(u^4(g_{\mathrm{E}} + h)) = 0$, for h compactly supported and small, the equation we have to solve is linear:

$$\Delta_{g_h} u - \frac{1}{8} R(g_h) u = 0.$$

•We want $u > 0$, and in the AE setting, we want $u \to 1$ at infinity. So let $u = 1 + v$. We want v to decay to 0 at infinity, and we set up function spaces to capture the decay of v.

• We re-write the PDE as $\Delta_{g_h} v - \frac{1}{8} R(g_h) v = \frac{1}{8} R(g_h)$.

• The operator $L = \Delta_{g_h} - \frac{1}{8} R(g_h)$ agrees with Δ outside a compact set. For small h, L is a small perturbation of Δ, which is an isomorphism on appropriate weighted spaces. Thus L is an isomorphism, and we can solve $Lv = \frac{1}{8} R(g_h)$, for v; because $R(g_h)$ is small, v will be small and tending to 0 at infinity.

Solving the constraints

Such metrics as we've just constructed are called harmonically flat: Consider an asymptotically flat metric with an end E with asymptotically flat coordinates $x = (x^i)$, so that for $|x| > r_0$, $g = u^4 g_{\mathrm{E}}$, $u \to 1$ at infinity, and $R(g) = 0$, i.e. $\Delta u = 0$.

Such u admits an expansion:

$$u(x) = 1 + \frac{A}{|x|} + \frac{\beta \cdot x}{|x|^3} + \cdots$$

where A is a constant and β is a vector. .

Note also that derivatives fall off one order faster: $\frac{\partial |x|^{-1}}{\partial x^i} = -|x|^{-3} x^i$, etc.
Exercise: $A = \frac{m}{2}$.
Such metrics are then asymptotic to Schwarzschild of mass m:
$g - g_S = O(|x|^{-2})$.
If $A \neq 0$, we define $c = \beta / A$: a straightforward computation shows
$\tilde{u}(y) := u(y + c) = 1 + \frac{A}{|y|} + O(|y|^{-3})$.

It makes sense then to identify c as the *center of mass*.

Solving the constraints

Now that we have some nice solutions (more on the way), let's get greedy, and start talking about the space of solutions.
For instance, suppose g is an AE solution to $R(g) = 0$ on $\mathbb{R}^3$. Is is close in some sense to a harmonically flat solution? (Density question) Yes! (Schoen-Yau 1980)

- Idea: For large θ, let χ_θ be a cutoff function that is 1 near $|x| = \theta$ and vanishes near $|x| = 2\theta$. Let $g_\theta = \chi_\theta g + (1 - \chi_\theta) g_{\mathrm{E}}$.
- Then $R(g_\theta)$ is supported in $\theta \leq |x| \leq 2\theta$, and in case $n = 3$ (and $q = 1$, say), $R(g_\theta) = O(|\theta|^{-3})$, which is small for θ large.
- Want: $0 < u \to 1$ at ∞, with $0 = R(u^4 g_\theta) = -8u^{-5}(\Delta_{g_\theta} u - \frac{1}{8} R(g_\theta) u)$.
- A similar analysis on the linear operator $\Delta_{g_\theta} - \frac{1}{8} R(g_\theta)$ as above allows us to conclude, for θ large enough, we can solve for $u > 0$, $u \to 1$ at infinity, which thus admits an expansion $u(x) = 1 + \frac{m}{2|x|} + \cdots$.
- It can be shown that the total mass does not change much under such approximation, and the metrics g and $u^4 g_\theta$ are close in a suitable topology for large θ.

Solving the constraints

For the full constraints, harmonically flat asymptotics are replaced by more general harmonic asymptotics. Let let $\widetilde{\mathcal{L}}_g X = \mathcal{L}_X g - \mathrm{div}_g(X) g$. Then outside a compact set in any end, we require $g = u^4 g_{\mathrm{E}}$, while

$$\pi_{ij} = u^2(\widetilde{\mathcal{L}}_{g_{\mathrm{E}}} X)_{ij} = X_{i,j} + X_{j,i} - (\sum_k X_{k,k})\delta_{ij}$$

in Cartesian coordinates for g_{E}. Here are virtues of this form:

• The vacuum constraints outside a compact set in components are

$$8\Delta u = u\big(- |\widetilde{\mathcal{L}} X|^2 + \frac{1}{2}(\mathrm{tr}(\widetilde{\mathcal{L}} X))^2\big)$$
$$\Delta X^i + 4u^{-1} u_{,j}(\widetilde{L} X)^j_i - 2u^{-1} u_{,i}\mathrm{tr}(\widetilde{\mathcal{L}} X) = 0$$

• $\pi_{ij} = -\frac{B^i x^j + B^j x^i}{|x|^3} + \sum_k \frac{B^k x^k}{|x|^3}\delta_{ij} + O(|x|^{-3})$, where $P^i = -\frac{B^i}{2}$ is the ADM linear momentum.

Solving the constraints

AE solutions to the vacuum constraints with harmonic asymptotics are dense in the space of all AE solutions, in a topology in which the ADM energy-momentum (E, P) is continuous.

On the other hand, it has been only very recently shown by F. C. Marques that the space of AE metrics of zero scalar curvature on $\mathbb{R}^3$ (and other topologies too) is connected, using Ricci flow with surgery.

Earlier results of Fischer and Marsden (cf. Bartnik, Chruściel-Delay) establish certain manifold structures on the space of solutions to the constraints.
A natural question is whether the space of solutions to the constraints, on a closed manifold, or on an AE manifold, say, admits an effective parametrization. An attempt at this is the conformal method of Lichnerowicz, Choquet-Bruhat, York...

The Conformal Method

- The ECE forms a system of 4 equations for 12 unknowns (locally, in dimension three).
- One might try to isolate certain parts of the data as “free” data—freely specifiable, giving parameters for the space of solutions to the constraints.
- The ECE will then give a system of PDE to determine the other parts of the data (g, K).
- This system of PDE will be a determined elliptic system.

The Conformal Method

The first step is the TT decomposition of symmetric two-tensors on (M, g). For simplicity, stick to dimension three, and M closed (or AE, say). We want to write a symmetric two-tensor Ψ as

$$\Psi = \Psi^{TT} + \Psi^{L} + \Psi^{Tr}$$

where Ψ^{TT} is TT (divergence-free, trace-free w.r.t. g), and Ψ^{L} is trace-free, and Ψ^{Tr} is a pure trace term:

$$\Psi^{Tr} = \frac{1}{3}(\mathrm{tr}_g \Psi) g.$$

Ansatz for Ψ^{L} (note that by design Ψ^{L} is trace-free):

$$\Psi^{L}_{ab} = (L_g W)_{ab} = W_{a;b} + W_{b;a} - \frac{2}{3}(\mathrm{div}_g W)_{ab}.$$

L_g is the conformal Killing operator; vectors in the kernel (CKV's) generate conformal isometries.

The Conformal Method

Therefore for any vector field W used to define Ψ^L, the tensor

$$\Psi^{TT} := \Psi - \Psi^L - \Psi^{Tr}$$

is symmetric and trace-free.
To deserve the "TT" superscript, then, we choose W to arrange $\text{div}_g(\Psi^{TT}) = 0$, obtaining an equation for W:

$$\text{div}_g(L_g W) = \text{div}_g(\Psi - \Psi^{Tr}).$$

Proposition

Given Ψ, we can solve the above equation for W, uniquely up to the addition of a CKV. Thus $\Psi^L = L_g W$ is uniquely determined.