2014 ESI Summer School
Mathematical Relativity: 99 Years of General Relativity
Problems from Constraint Equations Mini-course

Basic facts and conventions

- We use the Einstein summation convention: sum over a pair of upper and lower repeated indices.
- A comma in a subscript denotes partial differentiation, whereas a semicolon in a subscript denotes covariant differentiation. For example, if h is a $(1,2)$-tensor with components in a coordinate chart $h^i{}_{jk}$, then the covariant derivative ∇h is a $(1,3)$-tensor with components

$$h^i{}_{jk;\ell} := \nabla h\Big(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\Big) = (\nabla_{\frac{\partial}{\partial x^\ell}} h)\Big(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) = h^i{}_{jk,\ell} + \Gamma^i_{m\ell} h^m{}_{jk} - \Gamma^m_{j\ell} h^i{}_{mk} - \Gamma^m_{k\ell} h^i{}_{jm}.$$

- The Christoffel symbols for the connection compatible with a metric g (Riemannian or pseudo-Riemannian) are given by the rule (where $g^{i\ell} g_{\ell j} = \delta^i{}_j$)

$$\Gamma^k_{ij} = \frac{1}{2} g^{km}(g_{mj,i} + g_{im,j} - g_{ij,m}).$$

- In normal coordinates (x^i) about a point $p \in M$, we have $g_{ij,k}(p) = 0$ for all i, j and k, so that at p, $\Gamma^k_{ij}(p) = 0$ and $g^{ij}{}_{,k}(p) = (-g^{i\ell} g_{\ell m,k} g^{mj})(p) = 0$.
- Our convention for the Riemann curvature tensor is as follows:

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = R^\ell_{ijk} \frac{\partial}{\partial x^\ell}$$

$$R_{ijk\ell} = g_{m\ell} R^m_{ijk}.$$

- In our convention the Ricci tensor is then given by

$$\mathrm{Ric}(X,Y) = dx^i(R(\frac{\partial}{\partial x^i}, X, Y)) = g^{k\ell} g\left(R(\frac{\partial}{\partial x^k}, X, Y), \frac{\partial}{\partial x^\ell}\right)$$

$$R_{ij} = \mathrm{Ric}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = R^\ell_{\ell ij}.$$

- The scalar curvature is $R(g) = g^{ij} R_{ij}$. The Einstein tensor is $G = G(g) = \mathrm{Ric}(g) - \frac{1}{2} R(g) g$.
- Let g_{E} be the Euclidean metric on $\mathbb{R}^n$, with Cartesian coordinates $x = (x^1, \ldots, x^n)$ so that $g_{\mathrm{E}} = \delta_{ij} dx^i dx^j$, and let $|x| = \sqrt{\sum_{i=1}^n (x^i)^2}$. If $r = |x|$, $g_{\mathrm{E}} = dr^2 + r^2 g_{\mathbb{S}^{n-1}}$, where $g_{\mathbb{S}^{n-1}}$ is the standard round unit metric on the sphere $\mathbb{S}^{n-1}$. We recall the notation $g_{\mathbb{S}^2} = d\Omega^2 = d\phi^2 + \sin^2(\phi)\, d\theta^2$. Note that the angle convention is the mathematics convention, as opposed to the physics convention,
- If $\bar{g}$ is a Lorentzian or Riemannian metric on $\mathcal{S} = I \times M$, we let t be a global coordinate for I, and suppose in the Lorentzian case that $\frac{\partial}{\partial t}$ is everywhere time-like, so that in any case, each slice $M_t = \{t\} \times M$ is Riemannian, with induced metric $g(t)$. Let X be the orthogonal projection of $\frac{\partial}{\partial t}$ onto the tangent space of M_t at each (t,p). Then for each Y tangent to M_t, $\langle \frac{\partial}{\partial t}, Y \rangle = \langle X, Y \rangle$. If n is a unit normal field in the direction of $\frac{\partial}{\partial t}$, then there is a function $N > 0$ so that $\frac{\partial}{\partial t} = Nn + X$. Thus $\bar{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \langle n, n \rangle N^2 + |X|^2_g$. Then in local coordinates x^i for M, $\bar{g} = \langle n, n \rangle N^2 dt^2 + g_{ij}(dx^i + X^j dt) \otimes (dx^j + X^j dt)$.

Problems.

I. Mainardi and ADM equations and applications

1. Let $\mathcal{S} = I \times \mathbb{R}^3$, with $\bar{g} = -dt^2 + (a(t))^2 g_{\text{Eucl}}$. On the slices $\{t\} \times \mathbb{R}^3$, $g(t) = (a(t))^2 g_{\text{Eucl}}$, and $n = \frac{\partial}{\partial t}$ is a global unit normal field along the slices. Let a zero index denote the $\frac{\partial}{\partial t}$-component direction.

a. Use the ADM equations to compute the second fundamental form K, as well as the components of $\text{Ric}(\bar{g})$ in directions tangent to the slices.

b. Use the relation between g and K and $G_{\mu 0}$ discovered when we proved the constraint equations to compute $G_{\mu 0}$, and hence $G_{\mu\nu}$.

2. Consider $\mathcal{S} = I \times M$ and $\bar{g} = \langle n, n\rangle N^2 dt^2 + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt)$, as above.

a. For a vector field Y tangent to the slices M_t, show that $[n, Y] = (n[Y^i] + N^{-1} Y[X^i])\frac{\partial}{\partial x^i} + N^{-1} Y[N] n$.

b. Let $\hat{S}(X, Y) = \langle \overline{\nabla}_X n, Y\rangle = -\langle \overline{\nabla}_X Y, n\rangle = -K(X, Y)$. We recall $\overline{\nabla}_X Y = \nabla_X Y + \hat{K}(X, Y) n$, and that $\hat{S} = -\langle n, n\rangle \hat{K}$. The equation $\partial_t g_{ij} = 2n\hat{S}_{ij} + (\mathcal{L}_X g)_{ij}$ holds as in lecture. Recall that if T is a $(0, 2)$-tensor, then $T^2_{ij} = T_{ik} T_{\ell j} g^{k\ell}$.

Prove the *Mainardi equation*: for Y and Z vector fields tangent to the slices M_t,

$$\langle \bar{R}(Y, n, n), Z\rangle = -(\overline{\nabla}_n \hat{S})(Y, Z) - \hat{S}^2(Y, Z) - \langle n, n\rangle N^{-1} \text{Hess}_g N(Y, Z).$$

c. Suppose ν is a smooth unit normal field to a hypersurface $\Sigma \subset (\widehat{\Sigma}^n, h)$, and that $S_p(X) = (\nabla_X \nu)_p \in T_p\Sigma$ is the corresponding *shape operator*, the $(1, 1)$-tensor associated to the second fundamental form $\hat{S}$ of Σ. For each p in Σ, let γ_p be the geodesic with $\gamma_p'(0) = \nu(p)$. We note that given coordinates $x' = (x^1, \ldots, x^{n-1})$ for the hypersurface Σ, we let $(x^1, \ldots, x^n) \mapsto \gamma_{p(x')}(x^n) = \exp_{p(x')}(x^n \nu) \in M$. It's not hard to show that this map gives local coordinates for M, and that along Σ, $\frac{\partial}{\partial x^n} = \nu$.

Prove the Riccati equation $\nabla_\nu S + S^2 = -R(\cdot, \nu, \nu)$, where $(\nabla_\nu S)(X) = \nabla_\nu(S(X)) - S(\nabla_\nu X)$.

d. Recall that the mean curvature vector field $\mathbf{H}$ along Σ is independent of (local) orientation, and is given as the normal vector $\mathbf{H} = \text{tr}_\Sigma(\mathbb{II})$. Note that $\langle \mathbf{H}, n\rangle = -\text{tr}_\Sigma(S)$. Consider now a variation $F : I \times \Sigma \to M$ given by $\frac{\partial F}{\partial t}\big|_{(t,p)} = \nu(t, p)$, where $\nu(t, p)$ is a unit normal to the surface $\Sigma_t = F(t, \Sigma)$. Show that the variation in the mean curvature is given by $\partial_t(\text{tr}_\Sigma(S)) = -\|A\|^2 - \text{Ric}(\nu, \nu)$, where $K(X, Y) = g(X, -\nabla_X \nu) = g(\nabla_X Y, \nu)$ is the second fundamental form of Σ. Let $A(t) = \text{Area}(\Sigma_t)$. Compute $A'(t)$, and assuming $A'(0) = 0$, compute $A''(0)$.

II. Linearization of the scalar curvature map and static potentials.

3. Let $R(g) = g^{ij} R_{ij}$ be the scalar curvature of a metric (not necessarily Riemannian). Consider a variation $g(\epsilon) = g + \epsilon h$ of g in the direction of a symmetric $(0, 2)$-tensor field h (more generally, note that all you will use is that $g(\epsilon)$ is a metric smooth in t, with $g(0) = g$ and $g'(0) = h$).

Assume that for small $|\epsilon|$, $g(\epsilon)$ is a metric, as would be the case for h compactly supported. Define $L_g(h) := DR_g(h) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} R(g(\epsilon))$.

a. Derive the scalar curvature formula

$$R(g) = g^{ij}R_{ij} = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{k\ell}\Gamma^\ell_{ij} - \Gamma^k_{j\ell}\Gamma^\ell_{ik}\right).$$

b. Verify that the difference $S(X,Y) := \nabla_X Y - \widetilde{\nabla}_X Y$ defines a vector-valued $(0,2)$-tensor (i.e. a $(1,2)$ tensor $\widehat{S}(\theta, X, Y) = \theta(S(X,Y))$). Thus $\dot{\Gamma}^k_{ij} := \frac{d}{d\epsilon}\Big|_{\epsilon=0} \Gamma^k_{ij}$ form the components $(\delta\Gamma)^k_{ij}$ of a $(1,2)$-tensor $(\delta\Gamma)$. Argue that $\dot{\Gamma}^k_{ij} = \frac{1}{2}g^{km}(h_{mj;i} + h_{im;j} - h_{ij.m})$, where the covariant derivative is taken with respect to $g(0)$. (Hint: use $g(0)$-normal coordinates at p.)

c. Use the preceding part to aid in verifying the identities $\frac{d}{d\epsilon}\Big|_{\epsilon=0} R_{ij} = (\delta\Gamma)^k_{ij;k} - (\delta\Gamma)^k_{ik;j}$, and then

$$L_g(h) = -\Delta_g(\mathrm{tr}_g(h)) + \mathrm{div}_g(\mathrm{div}_g(h)) - \langle h, \mathrm{Ric}(g)\rangle_g$$

where the inner product of two $(0,2)$-tensors S and T is given by $\langle S, T\rangle = S_{ij}T_{k\ell}g^{ik}g^{j\ell}$, for example $\mathrm{tr}_g(S) = \langle g, S\rangle$.

d. Suppose that $(M, g, \pi = 0)$ solves the time-symmetric vacuum constraints $\Phi(g,\pi) = 0$, where $\Phi(g,\pi) = (R(g) - \|\pi\|^2_g + \frac{1}{2}(\mathrm{tr}_g(\pi))^2, \mathrm{div}_g(\pi))$. Here π^{ij} is a $(2,0)$-tensor, whose divergence defines a vector field.

Find $D\Phi_{(g,0)}(h,\sigma)$, where h is a symmetric $(0,2)$ tensor and σ is a symmetric $(2,0)$ tensor, each of compact support, and then find $D\Phi^*_{(g,0)}(N, X)$ by integration by parts. Since $\pi = 0$, the linearization simplifies dramatically from the general case. Compare your answer to that derived in class. If you want to try your hand at the general case, find a Starbucks and see if they serve a coffee larger than a Venti (sometimes they have a Trenta!).

4. Suppose (M, g) is Riemannian.

a. Suppose that $L^*_g N = 0$, and that γ is a unit-speed geodesic in (M^n, g). Let $h(t) = N(\gamma(t))$. Prove that $h(t)$ satisfies a second-order linear ODE. What does this say about the dimension of the kernel of L^*_g?

b. Suppose that $L^*_g N = 0$, but that f is not identically zero. Show that $\Sigma = N^{-1}(0)$ is a regular hypersurface, which is totally geodesic (zero second fundamental form). Hint: If $p \in \Sigma$ and $dN_p = 0$, what does part a. have to say about things?

c. Suppose that (M^n, g) is a closed manifold with negative scalar curvature. Find the kernel of L^*_g.

d. Consider the metric $g = (n-2)^{-1}g_{\mathbb{S}^1} \oplus g_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Show that $N(t, \omega) = \sin t$ is a static potential for g.

e. Does every Ricci-flat metric have a static potential? What can you say in case a metric (M, g) on a closed manifold with zero scalar curvature has a static potential?

f. Let $N : M \to \mathbb{R}$ be a smooth function. Define the Lorentzian metric $\bar{g} = -N^2 dt^2 \oplus g$ on the space $\mathcal{S} = I \times \{p \in M : f(p) \neq 0\}$. Prove that for X, Y tangent to M at p with $N(p) \neq 0$, we have

$\mathrm{Ric}(\bar g)(X,Y) = \mathrm{Ric}(g)(X,Y) - \frac{1}{N(p)}\mathrm{Hess}_g N(p)$, $\mathrm{Ric}(\bar g)(X, \frac{\partial}{\partial t}) = 0$, and $\mathrm{Ric}(\bar g)(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = N\Delta_g N$.

g. Conclude from part a. that a function N on M is a nontrivial element of the kernel of L_g^* if and only if the metric $\bar g$ as above is an Einstein metric. (Note that in the preceding problem you said something about the set $\{p \in M : N(p) = 0\}$ where the metric $\bar g$ may have issues.)

III. Conformal changes of metric.

5. a. Suppose (M^n, g) is a Riemannian metric and $\tilde g = e^\varphi g$. Show that

$$R(\tilde g) = e^{-\varphi}\left(R(g) - (n-1)\Delta_g\varphi - \frac{1}{4}(n-1)(n-2)|\nabla\varphi|_g^2\right).$$

b. In case $n \geq 3$, if we write $e^\varphi = u^{\frac{4}{n-2}}$ for $u > 0$, then

$$R(\tilde g) = u^{-\frac{n+2}{n-2}}\left(R(g)u - \frac{4(n-1)}{(n-2)}\Delta_g u\right).$$

c. Suppose M is compact with empty boundary. Let $c(n) = \frac{n-2}{4(n-1)}$. Let $L_g u = \Delta_g u - c(n)R(g)u$, the *conformal Laplacian.* Show that the total scalar curvature of $\tilde g = u^{\frac{4}{n-2}} g$ is given by

$$\int_M R(\tilde g)\, dv_{\tilde g} = c(n)^{-1}\int_M \left(|\nabla u|_g^2 + c(n)R(g)u^2\right) dv_g.$$

Hint: Show that $dv_{\tilde g} = u^{\frac{2n}{n-2}}\, dv_g$.

6. Recall the conformal Killing operator L is related to the Lie derivative $\mathcal{L}$ by the relationship $L_g W = \mathcal{L}_W g - \frac{2}{n}\mathrm{div}_g W g$.
a. Suppose (M, g_0) is three-dimensional. If $\phi > 0$ on M and $g = \phi^4 g_0$, show that for any trace-free symmetric $(2,0)$-tensor Ξ^{ab},

$$(\mathrm{div}_g(\phi^{-10}\Xi))^a = \phi^{-10}(\mathrm{div}_{g_0}\Xi)^a$$

and

$$(L_g W)^{ab} = \phi^{-4}(L_{g_0}W)^{ab}.$$

Can you figure out what the analogous statements would be in higher dimensions?

b. Suppose (M, g_0) is three-dimensional, that $\phi > 0$ on M and that $g = \phi^4 g_0$. If σ is TT for g_0, let $K^{cd} = \phi^{-10}(\sigma^{cd} + (L_{g_0}W)^{cd}) + \frac{1}{3}\phi^{-4}g_0^{cd}\tau$. Show the following:

$$R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 = -8\phi^{-5}(\Delta_{g_0}\phi - \frac{1}{8}R(g_0)\phi) - \phi^{-12}|\sigma + L_{g_0}W|_{g_0}^2 + \frac{2}{3}\tau^2$$
$$(\mathrm{div}_g K - d(\mathrm{tr}_g(K)))_a = \phi^{-6}\mathrm{div}_{g_0}(L_{g_0}W)_a - \frac{2}{3}d\tau_a.$$

7. a. Show directly (and in one line) that if h is symmetric with compact support, and if $L_{g_{\mathrm{E}}}h \geq 0$, then $L_{g_{\mathrm{E}}}h = 0$.

b. Show by elementary methods that there exists an infinite-dimensional space of TT tensors on $(\mathbb{R}^3, g_{\mathrm{E}})$ *with compact support.* Such tensors automatically satisfy $L_{g_{\mathrm{E}}}h = 0$.

c. (Open problem): If you can construct a non-trivial symmetric TT tensor h on $\mathbb{R}^3$ with compact support *and* so that $|h|^2_{g_{\mathrm{E}}}$ depends only on $|x|$, please let me know—a nice paper would come out of it.

8. Recall the operator $(\widetilde{\mathcal{L}}_g(X))_{ij} = X_{i;j} + X_{j;i} - X^k_{\ ;k} g_{ij}$.

a. If γ is a metric on M^3, let $g = u^4\gamma$ and $\pi_{ij} = u^2(\widetilde{\mathcal{L}}_\gamma(X))_{ij}$ for $u > 0$. Compute the constraints map $\Phi(g, \pi) = (R(g) - |\pi|^2_g + (\mathrm{tr}_g\pi)^2, \mathrm{div}_g\pi)$.

b. If $\gamma = g_{\mathrm{E}}$, show that the vacuum constraints can be written, in a Cartesian coordinate system for the background g_{E}, as follows:

$$8\Delta u = u\big(- |\widetilde{\mathcal{L}}X|^2 + \frac{1}{2}(\mathrm{tr}(\widetilde{\mathcal{L}}X))^2\big)$$
$$\Delta X^i + 4u^{-1}u_{,j}(\widetilde{L}X)^j_i - 2u^{-1}u_{,i}\mathrm{tr}(\widetilde{\mathcal{L}}X) = 0$$

c. If the above equations in part b. hold on an asymptotic end, one can show that u and X have partial expansions $u(x) = 1 + \frac{A}{|x|} + O(|x|^{-2})$, $X^i(x) = \frac{B^i}{|x|} + O(|x|^{-2})$, along with fall off for derivatives. Show that $\pi_{ij} = -\frac{B^i x^j + B^j x^i}{|x|^3} + \sum\limits_k \frac{B^k x^k}{|x|^3}\delta_{ij} + O(|x|^{-3})$, and that $P^i = -\frac{B^i}{2}$ is the ADM linear momentum.

d. (Open problem): Give some geometric framework for the operator $\widehat{\mathcal{L}}_g$. What is the "best" *Ansatz* for the asymptotic form of π_{ij}?

IV. Schwarzschild Geometry Basics Recall the spatial Schwarzschild metric $g_S = \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathrm{E}}$, defined on the manifold M given by $M = \mathbb{R}^3 \setminus \{0\}$ for $m > 0$, $M = \mathbb{R}^3$ for $m = 0$, and $M = \{x \in \mathbb{R}^3 : |x| > -\frac{m}{2}\}$ for $m < 0$.

9. a. We saw that $R(g_S) = 0$. Find $\mathrm{Ric}(g_S)$, which doesn't vanish.

b. Show that

$$m = \frac{1}{16\pi} \lim_{r\to+\infty} \int_{|x|=r} \sum_{i,j=1}^{3} \left((g_S)_{ij,i} - (g_S)_{ii,j}\right) \nu_e^j \, d\sigma_e$$

where the computation is done in the coordinates (x^1, x^2, x^3), and where ν_e is the Euclidean outward unit normal, and $d\sigma_e$ is the Euclidean area measure (where (x^i) are Cartesian coordinates for the Euclidean metric).

c. When $m < 0$, $A(r) \to 0$ as $r \to -(\frac{m}{2})^+$. Show that a radial geodesic from $r = r_0 > -\frac{m}{2}$ to $r = -\frac{m}{2}$ has finite length. Can the Schwarzschild metric with $m < 0$ be completed by adding in a point?

d. Let g_S be a Schwarzschild metric of non-zero mass m. Show that there is a one-dimensional kernel for $L^*_{g_S}$. Do this by showing first that for any function in the kernel, $\mathrm{Hess}_{g_S}(f) = f\mathrm{Ric}(g_S)$. Write this out in coordinates for which $g_S = (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\varphi^2 + \sin^2\varphi \; d\theta^2)$. Show that $\partial_\theta f = 0$ and $\partial_\varphi f = 0$, and then solve the remaining ODE for f.

e. Let $m > 0$. Find an isometric embedding of (M, g_S) into Euclidean space $\mathbb{E}^4$, identified in Cartesian coordinates (x, y, z, w) with $(\mathbb{R}^4, dx^2 + dy^2 + dz^2 + dw^2)$. It might be easiest to use the other coordinates we introduced for the Schwarzschild metric: $(1 - \frac{2m}{r})^{-1} dr^2 + r^2 g_{\mathbb{S}^2}$, $r > 2m$. (This corresponds to "half" of (M, g_S). The map you get will then extend by reflection to the other "half.") For $\omega \in \mathbb{S}^2$, look for an embedding of the form $x = r\omega \mapsto (r\omega, \xi(r)) \in \mathbb{R}^4$. Explain how this justifies the picture we've drawn of the Schwarzschild spatial slice.

f. When $m < 0$ the argument breaks in part e. down. Instead, look for an isometric embedding into Minkowski space $\mathbb{M}^4$, which is identified with $\mathbb{R}^4$ with the metric $dx^2 + dy^2 + dx^2 - dw^2$.

10. Let ∇ be the connection on (M, g_S), and for vector fields X and Y tangent to a surface $\Sigma \subset M$, let $\mathbb{I}(X, Y) = (\nabla_X Y)^{\mathrm{Nor}}$, and let $\mathbf{H} = \mathrm{tr}_\Sigma(\mathbb{I})$

a. For $m > 0$, show that $r \mapsto \frac{m^2}{4r}$ induces an isometry of g_S which fixes $\Sigma_0 = \{r = \frac{m}{2}\}$.

b. For $m > 0$, show that Σ_0 is totally geodesic in M. Express m in terms of the area of Σ_0.

c. Find the area $A(r)$ of $S_r = \{x : |x| = r\}$ of S_r in the metric g_S. For $m > 0$, show that $A(r)$ has a global minimum at $r = \frac{m}{2}$.

d. Fix r and find the second fundamental form and the mean curvature vector $\mathbf{H}$ of $S_r = \{x : |x| = r\}$ in the metric g_S.

e. Compare $A'(r)$ to $\int\limits_{S_r} \mathbf{H} \cdot \mathbf{X} \, d\sigma$, where $\mathbf{X} = \frac{\partial}{\partial r}$ and $d\sigma$ is the area measure induced by g_S.

f. The *Hawking mass* of a surface Σ is given by

$$m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int\limits_\Sigma H^2 \, d\sigma \right).$$

Find $m_H(S_r)$.

V. Mass and Center of Mass.

11. Suppose $(\mathbb{R}^3 \setminus \overline{B_{r_0}(0)}, g)$ is harmonically flat: $g = u^4 g_E$, $R(g) = 0$, i.e. $\Delta_{g_E} u = 0$, with $u(x) \to 1$ as $|x| \to +\infty$. We saw the expansion $u(x) = 1 + \frac{A}{|x|} + \frac{\beta_i x^i}{|x|^3} + O(|x|^{-3})$ via spherical harmonics.

a. Let $y = x + c$, for $c \in \mathbb{R}^n$. For $|y - c| > r_0$, find the asymptotic expansion of u as a function of y. Show that there is a chose of $c \in \mathbb{R}^3$ for which $u(y) = 1 + \frac{A}{|y|} + O(|y|^{-3})$.

b. Compute $\lim\limits_{r \to +\infty} \int\limits_{|x|=r} x^k \sum\limits_{i,j=1}^{3} (g_{ij,i} - g_{ii,j}) \, \nu_e^j \, dA_e$ where $\nu_e^j = \frac{x^j}{r}$. (Warning: this gives the center of mass, but the flux integral isn't the right form for more general asymptotically flat metrics.)

c. For $r_1 > r_0$, express $\int\limits_{r_1 \le |x| \le r} x^k \sum\limits_{i,j=1}^{3} (g_{ij,ij} - g_{ii,jj}) \; dx$ as a difference of two flux integrals, plus an "error term"—be careful—why is it an "error term"? More generally, for g asymptotically flat, with $R(g) \in L^1(M, g)$, what additional condition might you impose on g to show that this term is of smaller magnitude than the flux integrals?

12. Assume that h is a (smooth) transverse-traceless tensor at the Euclidean metric on $\mathbb{R}^3$. Let's use Cartesian coordinates x, so that covariant derivative components are computed via partial derivatives (the Christoffel symbols vanish). So $0 = \mathrm{tr}_{g_E} h = \sum_{i=1}^{3} h_{ii}$, and $0 = (\mathrm{div}_{g_E} h)_j = \sum_{i=1}^{3} h_{ij,j}$. Now, assume that h has compact support. Let $\gamma_\epsilon = g_E + \epsilon h$, and for $|\epsilon|$ sufficiently small, let $u_\epsilon > 0$ be the associated conformal factor so that with $g_\epsilon = u_\epsilon^4 \gamma_\epsilon$, $R(g_\epsilon) = 0$, and u_ϵ tends to 1 at infinity. Near infinity each u_ϵ is harmonic, and as such has an asymptotic expansion $u_\epsilon = 1 + \frac{m(\epsilon)}{2|x|} + O(|x|^{-2})$.

a. Show that $16\pi m(\epsilon) = -\int_{\mathbb{R}^3} R(\gamma_\epsilon) u_\epsilon \, d\mu_{g_\epsilon}$.

b. Show that $m'(0) = 0$ and that $16\pi m''(0) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_{g_E} h|^2 \, dx$.