2013 Graduate Mini-Course, NTU
Scalar curvature and the Einstein constraint equations
Exercises

We use the Einstein summation convention to sum over a pair of upper and lower repeated indices. Our convention for the Riemann curvature tensor agrees with that of John M. Lee's book, for instance (but is opposite in sign from that used in DoCarmo or O'Neill)—also watch the index convention:

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = R_{ijk}^{\ell} \frac{\partial}{\partial x^\ell}, \qquad R_{ijk\ell} = g_{m\ell} R_{ijk}^m.$$

From the definition of curvature, we immediately get the vector field version of the Ricci formula: $Z^i_{;jk} - Z^i_{;kj} = Z^\ell R^i_{kj\ell}$. The Ricci tensor in DoCarmo and Lee agree, which means the way they are defined from the Riemann tensor is slightly different to account for sign. In our convention,

$$\mathrm{Ric}(X,Y) = dx^i(R(\frac{\partial}{\partial x^i}, X, Y)) = g^{k\ell} g\left(R(\frac{\partial}{\partial x^k}, X, Y), \frac{\partial}{\partial x^\ell}\right)$$
$$R_{ij} = \mathrm{Ric}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = R^\ell_{\ell ij}.$$

Also, recall that a comma in a subscript denotes partial differentiation (with respect to some coordinate), whereas a semicolon in a subscript denotes covariant differentiation. For example, if T is a $(1,2)$-tensor with components T^i_{jk} in a coordinate chart, then the covariant derivative ∇T is a $(1,3)$-tensor with components

$$T^i_{jk;\ell} := \nabla T\Big(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\Big) = (\nabla_{\frac{\partial}{\partial x^\ell}} T)\Big(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) = T^i_{jk,\ell} + \Gamma^i_{m\ell} T^m_{jk} - \Gamma^m_{j\ell} T^i_{mk} - \Gamma^m_{k\ell} T^i_{jm},$$

where we recall the Christoffel symbols for the Levi-Civita connection are given by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}, \qquad \Gamma^k_{ij} = \frac{1}{2} g^{km} \left(g_{im,j} + g_{jm,i} - g_{ij,m}\right).$$

Please let us know if you think there is a typo or sign error!

Elementary warmup problems (you can skip to the next section if you've seen these).

Problem 1. If (M,g) is a Riemannian metric with Levi-Civita connection ∇. The *Hessian* of u is defined by $\mathrm{Hess}_g u = \nabla(du)$, and as such it is a $(0,2)$-tensor. The *Laplacian* is the trace of the Hessian: $\Delta_g u = \mathrm{tr}_g(\mathrm{Hess}_g u)$.

a. Show that $\mathrm{Hess}_g u(X,Y) = Y[X[u]] - \nabla_Y X[u]$, where $X[u] = du(X)$ is the directional derivative of u in the direction X. Conclude that the Hessian is symmetric.

b. Let $\det g = \det(g_{ij})$. Show that $\Delta_g u = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g}\, \frac{\partial u}{\partial x^j}\right)$.

c. Suppose $\Delta_g u = -\lambda u$ for some nontrivial (smooth) function u on a closed (compact, empty boundary) Riemannian manifold (M,g). Show that $\lambda \geq 0$. In case $\lambda = 0$, what is u?

d. If ∇ and $\widehat{\nabla}$ are two connections on M. Show that $S(X,Y) = \nabla_X Y - \widehat{\nabla}_X Y$ is tensorial in both X and Y (i.e. it is C^∞-linear in X and Y).

e. Prove the Ricci formula: if α is a one-form, then $\alpha_{i;jk} - \alpha_{i;kj} = \alpha_\ell R^\ell_{jki}$.

f. Use the Ricci formula to prove the following, for smooth functions u, where $\mathrm{grad}_g u$ is the vector metrically equivalent to du, i.e. $du(X) = g(\mathrm{grad}_g u, X)$:

$$g(\mathrm{grad}_g(\Delta_g u), \nabla u) + |\mathrm{Hess}_g u|_g^2 + \mathrm{Ric}_g(\mathrm{grad}_g u, \mathrm{grad}_g u) = \frac{1}{2}\Delta_g(|\mathrm{grad}_g u|_g^2).$$

Not-so-elementary warmup problems. The following formulas in the next two problems are essential. Those of you who haven't derived these should do so as soon as you can. You'll go through at least three Starbucks lattés while doing so, depending on how good you are at tensor calculations, and whether you ordered a grande or venti size. You could also accept these for now, and move on to the next section.

Problem 2. Linearization of Scalar Curvature. Let $R(g) = g^{ij}R_{ij}$ be the scalar curvature of a metric (not necessarily Riemannian). Consider a variation $g(t) = g + th$ of g in the direction of a symmetric $(0,2)$-tensor h (more generally, note that all you will use is that $g(t)$ is smooth in t, with $g(0) = g$ and $g'(0) = h$). For small t, $g(t)$ is a metric. Define $L_g(h) := DR_g(h) = \frac{d}{dt}\Big|_{t=0} R(g(t))$.

Derive the scalar curvature formula

$$R(g) = g^{ij}R_{ij} = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{k\ell}\Gamma^\ell_{ij} - \Gamma^k_{j\ell}\Gamma^\ell_{ik}\right)$$

and use it to verify the identity

$$L_g(h) = -\Delta_g(\mathrm{tr}_g(h)) + \mathrm{div}_g(\mathrm{div}_g(h)) - \langle h, \mathrm{Ric}(g)\rangle_g.$$

Hints: One approach is to compute in normal coordinates at a point p, so that $g_{ij}(p) = \delta_{ij}$, as well as $\Gamma^k_{ij}(p) = 0$, equivalently, $g_{ij,k}(p) = 0$. Indicated below are some formulas to guide you, and which *you should verify* as you compute. We emphasize here and below that all quantities are evaluated at p. In such normal coordinates, we have *at p*: $h_{ij;k\ell} = h_{ij,k\ell} - h_{mj}\Gamma^m_{ik,\ell} - h_{im}\Gamma^m_{jk,\ell}$. From this, one shows that *at p*,

$$-\Delta_g(\mathrm{tr}_g(h)) = -g^{k\ell}g^{ij}h_{ij;k\ell} = -\sum_{i,k}(h_{ii,kk} - 2h_{km}\Gamma^m_{ik,i})$$

$$\mathrm{div}_g(\mathrm{div}_g(h)) = g^{j\ell}g^{ik}h_{ij;k\ell} = \sum_{i,k}(h_{ik,ik} - h_{km}\Gamma^m_{ii,k} - h_{km}\Gamma^m_{ik,i}).$$

To find the variation of the scalar curvature, express the scalar curvature in terms of Christoffel symbols, and take a time derivative, and expand. Recall that if $A(t)$ is a smooth curve in $GL(n)$, then $\frac{d}{dt}A^{-1}(t)$ is easily computed from $A(t)A^{-1}(t) = I$ using the product rule.

Another approach is to note that $\frac{d}{dt}\Big|_{t=0}\Gamma^k_{ij}$ are the components $(\delta\Gamma)^k_{ij}$ of a tensor $\delta\Gamma$ (this follows from #1d on the above set of elementary problems), so that clearly the variation of the Ricci tensor is given by $\frac{d}{dt}\Big|_{t=0} R_{ij} = (\delta\Gamma)^k_{ij;k} - (\delta\Gamma)^k_{ik;j}$. One should now express $\delta\Gamma$ in terms of the covariant derivative of h.

Problem 3. Conformal deformation of scalar curvature.

a. Suppose (M^n, g) is a Riemannian metric and $\tilde{g} = e^{\varphi} g$. Show that

$$R(\tilde{g}) = e^{-\varphi}\left(R(g) - (n-1)\Delta_g \varphi - \frac{1}{4}(n-1)(n-2)|\nabla\varphi|_g^2\right).$$

b. In case $n \geq 3$, if we write $e^{\varphi} = u^{\frac{4}{n-2}}$ for $u > 0$, then

$$R(\tilde{g}) = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}}\left(\Delta_g u - \frac{n-2}{4(n-1)} R(g) u\right).$$

c. Let $c(n) = \frac{n-2}{4(n-1)}$ and $L_g u = \Delta_g u - c(n)R(g)u$ is the *conformal Laplacian*, show that the total scalar curvature of $\tilde{g} = u^{\frac{4}{n-2}} g$ is given by

$$\int_M R(\tilde{g})\, dv_{\tilde{g}} = c(n)^{-1} \int_M \left(|\nabla u|_g^2 + c(n)R(g)u^2\right) dv_g.$$

Hint: Show that $dv_{\tilde{g}} = u^{\frac{2n}{n-2}}\, dv_g$.

Problems I.

Problem 4. Consider the following metric with corners on $\mathbb{R}^3$ (refer to Professor Miao's first lecture): for $m \neq 0$, let

$$g_{ij}(x) = \begin{cases} \lambda^4 \delta_{ij}\ , & |x| \leq |m| \\ \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}\ , & |x| \geq |m| \end{cases},$$

where $\lambda = 1 + \frac{m}{2|m|} > 0$. Let $n = \lambda^{-2}\frac{\partial}{\partial r}$, where $r = |x|$ is the Euclidean distance, be the g-unit outward normal pointing to the boundary of the ball $B = \{x : |x| \leq m\}$. Let ∇^{λ} be the connection on $\mathbb{R}^3$ for the metric $(g_{\lambda})_{ij} = \lambda^4 \delta_{ij}$, and let ∇^m be the connection for $\left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}$ on $\{x : |x| > \frac{|m|}{2}\}$. Let E_1, E_2 be a local orthonormal frame for ∂B with respect to either metric (they agree on ∂B). The mean curvature H_- is defined as $\sum_{i=1}^{2} g(\nabla^{\lambda}_{E_i} n, E_i)$, and $H_+ = \sum_{i=1}^{2} g(\nabla^{m}_{E_i} n, Ei)$. Can you compare H_- and H_+? How does it depend on the sign of m?

Problem 5. Suppose ν is a smooth unit normal field to a hypersurface $\Sigma \subset (M^n, g)$, and that $S_p(X) = (\nabla_X \nu)_p \in T_p\Sigma$ is the corresponding *shape operator*. We note that given coordinates for the hypersurface Σ, $x' = (x^1, \dots, x^{n-1}) \mapsto p(x') \in \Sigma$, we let $(x^1, \dots, x^n) \mapsto \exp_{p(x')}(x^n \nu) \in M$. It's not hard to show that this map gives local coordinates for M, and that along Σ, $\frac{\partial}{\partial x^n} = \nu$.

a. Prove the Riccati equation $\nabla_\nu S + S^2 = -R(\cdot, \nu, \nu)$.
Note that in this equation, $(\nabla_\nu S)(X) = \nabla_\nu(S(X)) - S(\nabla_\nu X)$.

b. Consider now a variation $F : I \times \Sigma \to M$ given by $\frac{\partial F}{\partial t}\big|_{(t,p)} = \nu(t,p)$, where $\nu(t,p)$ is a unit normal to the surface $\Sigma_t = F(t, \Sigma)$. Show directly (without recourse to the second variation of area formula) that the variation in the mean curvature is given by $\frac{\partial H}{\partial t} = -\|A\|^2 - \mathrm{Ric}(\nu,\nu)$, where $A(X,Y) = g(X, -\nabla_X \nu) = g(\nabla_X Y, \nu)$ is the second fundamental form, and H is the trace of S

along Σ.

PROBLEM 6. Let M be a closed (compact, empty boundary) manifold. Let $\mathcal{M}$ be the space of smooth Riemannian metrics on M, let $\mathcal{M}_1$ be the space of smooth Riemannian metrics with unit volume, and given any Riemannian metric g, let $[g] = \{fg : f \in C^\infty(M),\ f > 0\}$ be the conformal class of g. Prove that a Riemannian metric $g \in \mathcal{M}_1$ is stationary for the Einstein-Hilbert action $\mathcal{R}(g) = \int_M R(g)\, dv_g$ amongst variations $g_t \in [g] \cap \mathcal{M}_1$ ($t \in (-\epsilon, \epsilon)$, $g_0 = g$) if and only if g has constant scalar curvature.

REMARK: One can equivalently consider $g \in \mathcal{M}$ stationary for $\overline{\mathcal{R}}(g) = \frac{\mathcal{R}(g)}{(\mathrm{vol}(g))^{1-\frac{2}{n}}}$ amongst variations $g_t \in [g]$.

PROBLEM 7. Recall the definition of the energy of an asymptotically flat initial data set:

$$E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \to +\infty} \int_{\{|x|=r\}} \sum_{i,j=1}^{n} (g_{ij,j} - g_{ii,j}) \frac{x^j}{|x|}\, dA_e.$$

a. Suppose $g_{ij} = \left(1 + \frac{A}{|x|^{n-2}} + O_2(|x|^{1-n})\right)^{\frac{4}{n-2}} \delta_{ij}$, where $f \in O_2(|x|^\gamma)$ means $|\partial_x^\alpha f(x)| = O(|x|^{\gamma - |\alpha|})$ for $|\alpha| \le 2$. Show that $E = 2A$.

REMARK: Recall the Riemannian Schwarzschild metric $(g_S)_{ij} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij}$. The above calculation shows that the energy is precisely m.

b. Generalize part a.: If g is asymptotically flat with energy $E(g)$, say $(g_{ij} - \delta_{ij}) = O_2(|x|^{-q})$ for $q \in \left(\frac{n-2}{2}, n-2\right]$. Let $u = 1 + \frac{A}{|x|^{n-2}} + O_2(|x|^{1-n}) > 0$. Show that the energy of $\bar{g} = u^{\frac{4}{n-2}} g$ satisfies $E(\bar{g}) = E(g) + 2A$. If $0 < u \le 1$, then $E(\bar{g}) \le E(g)$.

c. Suppose $g_{ij}(x) = \left(C_1 + \frac{C_2}{|x|^{n-2}} + O_2(|x|^{1-n})\right)^{\frac{4}{n-2}} \delta_{ij}$. Show that g is asymptotically flat, and $E = 2C_1 C_2$.

PROBLEMS II.

EUCLIDEAN HARMONIC FUNCTIONS

PROBLEM 8. a. Verify that the following distributional equations hold: $\Delta(\frac{1}{2\pi} \log|x|) = \delta_0$ in dimension $n = 2$, while $\Delta(\frac{1}{(2-n)n\omega_n}|x|^{2-n}) = \delta_0$ in dimensions $n > 2$. Here δ_0 is the Dirac delta distribution at the origin.

b. Suppose $f \in C_c^2(\mathbb{R}^n)$, $n > 2$. Suppose $\mathrm{spt}(f) \subset \{x : |x| \le K\}$. Then if we let $u(x) = \frac{1}{(2-n)n\omega_n} \int_{\mathbb{R}^n} |x-y|^{2-n} f(y)\, dy$, then $\Delta u = f$ by the above. Moreover, show that u has an expansion of the form $u(x) = \frac{A}{|x|^{n-2}} + \frac{B_i x^i}{|x|^n} + O(|x|^{-n})$. Express the constants A and B_i in terms of integrals involving f.

Problem 9. a. Show that if u is harmonic with an isolated singularity at $x = 0$, then the singularity is in fact removable if $\lim_{x\to 0} |x|^{n-2}u(x) = 0$ in case $n > 2$, and in case $n = 2$, if $\lim_{x\to 0} \frac{u(x)}{\log|x|} = 0$.

b. If $K[u]$ is the Kelvin transform of u, find $\Delta(K[u])$ in terms of Δu. Conclude that $K[u]$ is harmonic if and only if u is harmonic. Recall $K[u](x) = |x|^{2-n}u(x^*)$, $x^* = |x|^{-2}x$.

c. Suppose $n > 2$, and v is harmonic outside a compact set. Show that $K[v]$ has a removable singularity at the origin if and only if $\lim_{|x|\to+\infty} v(x) = 0$. In this case we say v is *harmonic at infinity.*

Problem 10. If v is harmonic at infinity (cf. Problem 9) and $n > 2$, v admits an expansion at infinity in terms of spherical harmonics. Show in fact that $v(x) = \frac{a_0}{|x|^{n-2}} + \frac{a_i x^i}{|x|^n} + O(|x|^n)$, and derive the next order term, in case $n = 3$.

Schwarzschild Geometry Basics

For simplicity, we let $n = 3$. Let g_E be the Euclidean metric, with Cartesian coordinates $x = (x^1, x^2, x^3)$ so that $g_E = \delta_{ij}dx^i dx^j$, and let $|x| = \sqrt{\sum_{i=1}^{3}(x^i)^2}$. If $r = |x|$, $g_E = dr^2 + r^2\, g_{\mathbb{S}^2} = dr^2 + r^2(d\phi^2 + \sin^2(\phi)\, d\theta^2)$. Consider the spatial Schwarzschild metric $g_S = \left(1 + \frac{m}{2|x|}\right)^4 g_E$, defined on$\mathbb{R}^3 \setminus \{0\}$ for $m > 0$, on $\mathbb{R}^3$ for $m = 0$, and on $\{x \in \mathbb{R}^3 : |x| > -\frac{m}{2}\}$ for $m < 0$. Recall that a portion of the maximally extended Schwarzschild space-time $\mathcal{S}$ is given by

$$\bar{g}_S = -\left(\frac{1 - \frac{m}{2|x|}}{1 + \frac{m}{2|x|}}\right)^2 dt^2 + \left(1 + \frac{m}{2|x|}\right)^4 g_E$$

on $|x| > \frac{|m|}{2}$ in case $m \neq 0$. You may use the fact that $\mathrm{Ric}(\bar{g}_S) = 0$, and we let $M = \mathcal{S} \cap \{t = 0\}$, so that with the induced metric, M is (possibly a subset of) the Schwarzschild geometry defined above.

Problem 11. a. Show that M is totally geodesic in $\mathcal{S}$.

b. Show that $R(g_S) = 0$ by using the conformal deformation of scalar curvature recalled above. Show that this is consistent with the Einstein constraint equations.

c. Is $\mathrm{Ric}(g_S) = 0$?

Problem 12. a. For $m > 0$, show that $r \mapsto \frac{m^2}{4r}$ induces an isometry of g_S which fixes $\Sigma_0 = \{r = \frac{m}{2}\}$.

b. For $m > 0$, show that Σ_0 is totally geodesic in M. Express m in terms of the area of Σ_0.

c. Find the area $A(r)$ of $S_r = \{x : |x| = r\}$ of S_r in the metric g_S. For $m > 0$, show that $A(r)$ has a global minimum at $r = \frac{m}{2}$.

d. When $m < 0$, $A(r) \to 0$ as $r \to -(\frac{m}{2})^+$. Furthermore, a radial geodesic from $r = r_0 > -\frac{m}{2}$ to $r = -\frac{m}{2}$ has finite length. Can the Schwarzschild metric with $m < 0$ be completed by adding in a point?

Problem 13. a. Fix r and find the second fundamental form $I\!I$ and the mean curvature vector $\mathbf{H}$ of $S_r = \{x : |x| = r\}$ of S_r in the metric g_S.

b. Compare $A'(r)$ to $\int_{S_r} \mathbf{H} \cdot \mathbf{X}\, d\sigma$, where $\mathbf{X} = \frac{\partial}{\partial r}$ and $d\sigma$ is the area measure induced by g_S.

c. The *Hawking mass* of a surface Σ is given by

$$m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2\, d\sigma \right).$$

Find $m_H(S_r)$.

Problem 14. Show that there are no closed minimal surfaces in (M, g_S) other than Σ_0 as in Problem 12b. in case $m > 0$. (The argument should follow along the lines of the proof that there are no closed minimal surfaces in $(\mathbb{R}^3, g_E)$.)

Problem 15. Embedding the Schwarzschild spatial metric.

a. Let $m > 0$. Find an isometric embedding of (M, g_S) into Euclidean space $\mathbb{E}^4$, identified in Cartesian coordinates (x, y, z, w) with $(\mathbb{R}^4, dx^2 + dy^2 + dz^2 + dw^2)$. It might be easiest use the other coordinates we introduced for the Schwarzschild metric: $(1 - \frac{2m}{r})^{-1} dr^2 + r^2\, g_{\mathbb{S}^2}$, $r > 2m$. (This corresponds to "half" of (M, g_S). The map you get will then extend by reflection to the other "half.") For $\omega \in \mathbb{S}^2$, look for an embedding of the form $x = r\omega \mapsto (r\omega, \xi(r)) \in \mathbb{R}^4$. Explain how this justifies the picture we've drawn of the Schwarzschild spatial slice.

b. When $m < 0$ the argument breaks down. Instead, look for an isometric embedding into Minkowski space $\mathbb{M}^4$, which is identified with $\mathbb{R}^4$ with the metric $dx^2 + dy^2 + dx^2 - dw^2$.

Problems III.

Problem 16. On the center of mass. Suppose $(\mathbb{R}^3 \setminus \overline{B_{r_0}(0)}, g)$ is harmonically flat: $g = u^4 g_E$, $R(g) = 0$, i.e. $\Delta_{g_E} u = 0$, with $u(x) \to 1$ as $|x| \to +\infty$. We saw the expansion $u(x) = 1 + \frac{A}{|x|} + \frac{\beta_i x^i}{|x|^3} + O(|x|^{-3})$ via spherical harmonics.

a. Let $y = x + c$, for $c \in \mathbb{R}^n$. For $|y - c| > r_0$, find the asymptotic expansion of u as a function of y. Show that there is a choice of $c \in \mathbb{R}^3$ for which $u(y) = 1 + \frac{A}{|y|} + O(|y|^{-3})$.

b. Compute $\lim_{r \to +\infty} \int_{|x|=r} x^k \sum_{i,j=1}^{3} (g_{ij,i} - g_{ii,j})\, \nu_e^j\, dA_e$ where $\nu_e^j = \frac{x^j}{r}$. (Warning: this gives the center of mass, but the flux integral isn't the right form for more general asymptotically flat metrics.)

c. For $r_1 > r_0$, express $\int_{r_1 \le |x| \le r} x^k \sum_{i,j=1}^{3} (g_{ij,ij} - g_{ii,jj})\, dx$ as a difference of two flux integrals, plus an "error term"—be careful—why is it an "error term"? More generally, for g asymptotically flat, with $R(g) \in L^1(M, g)$, what additional condition might you impose on g to show that this term is of smaller magnitude than the flux integrals? Note that you might first show that for $g_{ij} - \delta_{ij} = O_2(|x|^{-q})$, $R(g) = \sum_{i,j} (g_{ij,ij} - g_{ii,jj}) + O(|x|^{-(2q+2)}$.

Problem 17. Variation of the mass. Consider $\mathbb{R}^3$ with the Euclidean metric g_E. Assume that h is a compactly supported smooth symmetric $(0,2)$-tensor on $\mathbb{R}^3$. For $t \in (-\epsilon, \epsilon)$, we let $\gamma_t = g_E + th$, which is still a metric for $\epsilon > 0$ sufficiently small. For ϵ sufficiently small, we can let $u_t > 0$ be the associated conformal factor so that with $g_t = u_t^4 \gamma_t$, $R(g_t) = 0$. Let $m(t)$ be the ADM mass of g_t. Since h has compact support, u_t is (Euclidean) harmonic near infinity, thus has a full spherical harmonic expansion.

a. Show that $16\pi m(t) = -\int_{\mathbb{R}^3} R(\gamma_t) u_t \, d\mu_{\gamma_t}$.

b. Show directly (without recourse to the Positive Mass Theorem) that $m'(0) = 0$.

c. Generalize the above in the following way: let $(\mathbb{R}^3, g)$ be asymptotically flat with $R(g) = 0$. Let $g_t = u_t^4 R(g + th)$ with $R(g_t) = 0$, with g_t asymptotically flat. Compute $m'(0)$. What happens in case $m'(0) = 0$—can you pick a particularly good choice for h?

Problem 18. Static potentials, I. Let L_g^* be the formal adjoint of L_g (the linearization of scalar curvature, see above) defined by integration by parts: for any smooth compactly supported symmetric $(0,2)$-tensor h, $\int_M h \cdot L_g^* f \, d\mu_g = \int_M f L_g h \, d\mu_g$. Clearly we have $L_g^* f = -(\Delta_g f) g + \mathrm{Hess}_g f - f \mathrm{Ric}(g)$. A nontrivial element in the kernel of L_g^* is called a *static potential.*

Let (M^n, g) be connected.

a. Suppose that $L_g^* f = 0$, and that γ is a unit-speed geodesic in (M^n, g). Let $h(t) = f(\gamma(t))$. Prove that $h(t)$ satisfies a second-order linear ODE. What does this say about the dimension of the kernel of L_g^*?

b. Suppose that $L_g^* f = 0$, but that f is not identically zero. Show that $\Sigma = f^{-1}(0)$ is a regular hypersurface, which is totally geodesic (zero second fundamental form). Hint: If $p \in \Sigma$ and $df_p = 0$, what does part a. have to say about things?

c. Suppose that (M^n, g) is a closed manifold with negative scalar curvature. Find the kernel of L_g^*.

d. Find the kernel of L_g^* for the following metrics g: (i) $(\mathbb{R}^n, g_E)$; (ii) $(\mathbb{T}^n, g_F)$ (a flat torus);
(iii) $(\mathbb{S}^n, g_{\mathbb{S}^n})$ (round unit sphere $\mathbb{S}^n$, which you can think of as sitting in $\mathbb{R}^{n+1}$).

e. Consider the metric $g = (n-2)^{-1} g_{\mathbb{S}^1} \oplus g_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Show that $f(t, \omega) = \sin t$ is a static potential for g.

f. Does every Ricci-flat metric have a static potential? What can you say in case a metric (M, g) on a closed manifold with zero scalar curvature has a static potential?

Problem 19. Static potentials, II.

a. Let $f : M \to \mathbb{R}$ be a smooth function. Define the metric $\bar{g} = -f^2 dt^2 \oplus g$ on the space $N = I \times \{p \in M : f(p) \neq 0\}$. Prove that for X, Y tangent to M at p with $f(p) \neq 0$, we have $\mathrm{Ric}(\bar{g})(X, Y) = \mathrm{Ric}(g)(X, Y) - \frac{1}{f(p)} \mathrm{Hess}_g f(p)$, $\mathrm{Ric}(\bar{g})(X, \frac{\partial}{\partial t}) = 0$, and $\mathrm{Ric}(\bar{g})(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = f \Delta_g f$.

b. Conclude from part a. that a function f on M is a nontrivial element of the kernel of L_g^* if and only if the metric $\bar{g}$ as above is an Einstein metric. (Note that in the preceding problem you said something about the set $\{p \in M : f(p) = 0\}$ where the metric $\bar{g}$ may have issues.)

c. Identify a static potential for the Schwarzschild metric. If you use conformally flat coordinates, where does the static potential have zeros? Did you already know some things about this special level set?

d. Let g_S be a Schwarzschild metric of non-zero mass m. Show that there is a one-dimensional kernel for $L_{g_S}^*$. Do this by showing first that for any function in the kernel, $\mathrm{Hess}_{g_S}(f) = f\mathrm{Ric}(g_S)$. Write this out in coordinates for which $g_S = (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\varphi^2 + \sin^2\varphi\, d\theta^2)$. Show that $\partial_\theta f = 0$ and $\partial_\varphi f = 0$, and then solve the remaining ODE for f.

Problem 20. A PDE problem for those who know some functional analysis and elliptic theory. Suppose (M^n, g) is a closed, smooth Riemannian manifold. Suppose L_g^* has trivial kernel.

a. Show that $L_g L_g^*$ has trivial kernel.

b. Show that L_g maps the space of smooth symmetric tensors on M onto the space of smooth functions, i.e. for any smooth function F on M, there is a smooth symmetric $(0,2)$-tensor h so that $L(h) = F$. In fact, find a smooth function u so that $h = L_g^* u$ solves the equation, i.e. $L_g L_g^*(u) = F$.

Hint: Consider the functional $\mathcal{G}(v) = \int_M \left(\frac{1}{2}|L_g^* v|^2 - Fv\right)\, d\mu_g$. Show that there is a unique minimizer $u \in H^2$, say, where $H^2 = H^2(M, g)$ is the Sobolev space of tensor fields which are in $L^2(M, g)$ along with two covariant derivatives. To do this, you'll need to prove there is a $C > 0$ so that for all $v \in H^2$, $\|v\|_{H^2} \leq C\|L_g^* v\|_{L^2}$. This will involve a pointwise estimate, together with the Rellich theorem. Why is the minimizer smooth?

Problem 21. A projected problem in finite dimensions. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator. Define $T^* : \mathbb{R}^m \to \mathbb{R}^m$ to be the adjoint, defined by (where angle brackets denote dot product in the domain or co-domain of T: for all $v \in \mathbb{R}^n$ and all $w \in \mathbb{R}^m$, $\langle Tv, w\rangle = \langle v, T^* w\rangle$.

a. Show that the image $\mathrm{Ran}(T)$ is the orthogonal complement of the kernel $\ker(T^*)$ of the adjoint: $\mathbb{R}^m = \mathrm{Ran}(T) \oplus \ker(T^*)$. The kernel of T^* is sometimes called the *co-kernel*.

b. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth, $f(0) = 0$, and $Df(0) : T_0\mathbb{R}^n \to T_0\mathbb{R}^m$ has rank r, with image $\mathrm{Ran}(Df(0)) = S \subset \mathbb{R}^m$. Let $\Pi_S : \mathbb{R}^m \to S$ be the orthogonal projection onto S. Let $F : \mathbb{R}^n \to S$ be the smooth function given by $F = \Pi_S \circ f$. Show that there is a neighborhood V of the origin in $\mathbb{R}^m$ and a neighborhood U of the origin in $\mathbb{R}^n$ so that $F(U) = V \cap S$.

Problem 22. A mass estimate via the Penrose Inequality. Suppose (M, g) is an asymptotically flat three-manifold with $R(g) \geq 0$, and suppose Σ is an outermost minimal sphere in M, with respect to an asymptotically flat end $\mathcal{E}$ of mass m. The *Penrose inequality* states that $m \geq \sqrt{\frac{|\Sigma|}{16\pi}}$. If the sectional curvatures of M are bounded above by a constant $C > 0$, give an estimate of m in terms of C.