Patterns, Permutations, and Placements

Jonathan S. Bloom

Dartmouth College

Wake Forest University - February 2014
Permutations

Definition

A permutation of length \( n \) is a rearrangement of the numbers \( 1, 2, \ldots, n \).

Notation

Let \( S_n \) denote the set of all permutations of length \( n \).

Example

\( S_3 = \{123, 132, 213, 231, 312, 321\} \), and \( |S_n| = n! \).
Permutations

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A permutation of length $n$ is a rearrangement of the numbers $1, 2, \ldots, n$. 

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Example
$S_3 = \{123, 132, 213, 231, 312, 321\}$, and $|S_n| = n!$.
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and

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and

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Stack Sorting

D. Knuth (1968) defined a sorting algorithm, called stack sorting.
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**Examples**

*Let* $\alpha = 3\ 1\ 2\ 5\ 4$
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Examples

Let $\alpha = 3 \ 1 \ 2 \ 5 \ 4$

Let $\pi = 3 \ 1 \ 4 \ 5 \ 2$
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Examples

Let $\alpha = 3 \ 1 \ 2 \ 5 \ 4$

\[\begin{array}{c}
3 \\
\end{array}\]

Let $\pi = 3 \ 1 \ 4 \ 5 \ 2$

\[\begin{array}{c}
3 \\
\end{array}\]
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Let $\alpha = 3 \ 1 \ 2 \ 5 \ 4$

$\pi = 3 \ 1 \ 4 \ 5 \ 2$ is NOT stack-sortable

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$\pi = 3\ 1\ 4\ 5\ 2$

$\pi$ is not stack-sortable.
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D. Knuth (1968) defined a sorting algorithm, called *stack sorting*.

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Examples

Let $\alpha = 3 1 2 5 4$

- $\alpha$ is stack-sortable
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Let \( \alpha = 3 \ 1 \ 2 \ 5 \ 4 \) ▶ \( \alpha \) is stack-sortable

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Examples

Let $\alpha = 3 \ 1 \ 2 \ 5 \ 4$

- $\alpha$ **is stack-sortable**

Let $\pi = 3 \ 1 \ 4 \ 5 \ 2$

```plaintext
1 2 3 4 5
```

```plaintext
1 2
```

```plaintext
5
4
3
```
Stack Sorting

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Examples

Let $\alpha = 3\ 1\ 2\ 5\ 4$

$\triangleright \alpha$ is stack-sortable

Let $\pi = 3\ 1\ 4\ 5\ 2$
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**Examples**

Let $\alpha = 3 \ 1 \ 2 \ 5 \ 4$

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- $\pi$ is not stack-sortable
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**Examples**

Let \( \alpha = 3\ 1\ 2\ 5\ 4 \)
- \( \alpha \) is stack-sortable

Let \( \pi = 3\ 1\ 4\ 5\ 2 \)
- \( \pi \) is NOT stack-sortable
Stack Sorting

Question

Why is $\alpha = 3 \ 1 \ 2 \ 5 \ 4$ stack-sortable, while $\pi = 3 \ 1 \ 4 \ 5 \ 2$ is NOT?

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Why is $\alpha = 3 \ 1 \ 2 \ 5 \ 4$ stack-sortable, while $\pi = 3 \ 1 \ 4 \ 5 \ 2$ is NOT?

Theorem (D. Knuth 1968)
$\pi$ is NOT stack-sortable $\iff$ $\pi$ has three entries whose relative ordering is “231”.

Examples
Question

Why is $\alpha = 3\ 1\ 2\ 5\ 4$ stack-sortable, while $\pi = 3\ 1\ 4\ 5\ 2$ is NOT?

Theorem (D. Knuth 1968)

$\pi$ is NOT stack-sortable $\iff$ $\pi$ has three entries whose relative ordering is “231”.

Examples

$\pi = 3\ 1\ 4\ 5\ 2$ is NOT stack-sortable
Stack Sorting

Question
Why is $\alpha = 3 \ 1 \ 2 \ 5 \ 4$ stack-sortable, while $\pi = 3 \ 1 \ 4 \ 5 \ 2$ is NOT?

Theorem (D. Knuth 1968)
$\pi$ is NOT stack-sortable $\iff$ $\pi$ has three entries whose relative ordering is “231”.

Examples

$\pi = 3 \ 1 \ 4 \ 5 \ 2$ is NOT stack-sortable
$\implies \pi \ contains \ the \ pattern \ 231$
Stack Sorting

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Theorem (D. Knuth 1968)
$\pi$ is NOT stack-sortable $\iff$ $\pi$ has three entries whose relative ordering is “231”.

Examples

$\pi = 3 \ 1 \ 4 \ 5 \ 2$ is NOT stack-sortable
$\Rightarrow \pi$ contains the pattern 231

$\alpha = 3 \ 1 \ 2 \ 5 \ 4$ is stack-sortable
Stack Sorting

Question

Why is $\alpha = 3 \ 1 \ 2 \ 5 \ 4$ stack-sortable, while $\pi = 3 \ 1 \ 4 \ 5 \ 2$ is NOT?

Theorem (D. Knuth 1968)

$\pi$ is NOT stack-sortable $\iff \pi$ has three entries whose relative ordering is “231”.

Examples

$\pi = 3 \ 1 \ 4 \ 5 \ 2$ is NOT stack-sortable

$\implies \pi$ contains the pattern 231

$\alpha = 3 \ 1 \ 2 \ 5 \ 4$ is stack-sortable

$\implies \alpha$ avoid the pattern 231
Permutation Patterns

It's easier with pictures!

$$\pi = 31452$$
Permutation Patterns

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\[ \pi = 31452 \]
Permutation Patterns

Its easier with pictures!

\[ \pi = 31452 \]

\[ \begin{array}{cccc}
\times & \\
\times & \\
\times & \\
\times & \\
\times & \\
\end{array} \]

\[ \Rightarrow \]

\[ \pi \text{ contains } 123 \]
Permutation Patterns

Its easier with pictures!

\[ \pi = 31452 \]

[Diagram of a permutation pattern with marked cells]

- \( \pi \) contains 123

Notation

Let \( S_n(\tau) \) be the set of permutations of length \( n \) that avoid \( \tau \).
Permutation Patterns

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\[ \pi = 31452 \]

\[ \begin{array}{ccc}
\times & \times & \times \\
\times & \times & \\
\times & & \\
\end{array} \]

- \( \pi \) contains 123
Permutation Patterns

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\[ \pi = 31452 \]

\[ \begin{array}{ccc}
\times & \times \\
\times & & \\
& & \times \\
& \times & \\
\end{array} \]

- \( \pi \) contains 123
- \( \pi \) contains 213
Permutation Patterns

It's easier with pictures!

\[ \pi = 31452 \quad \rightarrow \quad \begin{array}{cccc}
\times & \times & \times & \\
\times & \times & \times & \\
\times & \times & \times & \\
\end{array} \]

- \( \pi \) contains 123
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Notation

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\[ \pi = 31452 \]

\( \rightarrow \)

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\end{array} \]

\[ \begin{array}{c}
\text{\( \pi \) contains 123} \\
\text{\( \pi \) contains 213} \\
\text{\( \pi \) avoids 321} \\
\end{array} \]
Permutation Patterns

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\[ \pi = 31452 \]

\[ \begin{array}{c|c|c|c|c|c} 
\hline & & & & & \\
\hline & & & & & \\
\hline & & & & & \\
\hline & & & & & \\
\hline & & & & & \\
\hline \end{array} \]

- \( \pi \) contains 123
- \( \pi \) contains 213
- \( \pi \) avoids 321

Notation

Let \( S_n(\tau) \) be the set of permutations of length \( n \) that avoid \( \tau \).
Definition
We say two patterns $\tau, \sigma \in S_k$ are \textbf{Wilf-equivalent} provided

$$|S_n(\tau)| = |S_n(\sigma)|$$

for all $n$. 
Example (Patterns of length 2)

What permutations avoid 21?

\[ S^2 = \{12, 21\} \]

\[ S^3(21) = \{123\} \]

In general, \[ S^n(21) = \{123\ldots n\} \].

\[ S^n(12) = \{n\ldots 321\} \]

\[ 12 \text{ is Wilf-equivalent to } 21 \]
Permutation Patterns

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Patterns of length 3

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Patterns of length 3

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\[ |S_3(321)| = 5 \]
\[ |S_4(321)| = 14 \]
\[ |S_5(321)| = 42 \]
\[ \vdots \]
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\[ |S_n(321)| = \frac{1}{n+1} \binom{2n}{n} \]

In fact, this is true for ALL length 3 patterns!!
Patterns of length 3

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What permutations avoid 321?

$$S_3 = \{123, 132, 213, 231, 312, 321\}$$

$$\therefore S_3(321) = \{123, 132, 213, 231, 312\}$$

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$$|S_5(321)| = 42$$

$$\therefore \quad |S_n(321)| = \frac{1}{n+1} \binom{2n}{n} = nth \text{ Catalan number}$$

In fact, this is true for ALL length 3 patterns!!

\(\therefore\) ALL length 3 patterns are Wilf-equivalent
Patterns of length 4

Things get messy...
Patterns of length 4

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<tr>
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<th>6</th>
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⇒ NOT all patterns of length 4 are Wilf-equivalent.

What is known?
▶ I. Gessel (1990) gave a formula for $|S_n(1234)|$.
▶ M. Bóna (1997) gave a formula for $|S_n(2314)|$.

Open Problem
Find a formula for $|S_n(1324)|$. 
Patterns of length 4

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In fact, every pattern of length 4 is Wilf-equivalent to one of:

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<td>S_n(2314)</td>
<td>$</td>
<td>103</td>
<td>512</td>
<td>2740</td>
</tr>
<tr>
<td>$</td>
<td>S_n(1234)</td>
<td>$</td>
<td>103</td>
<td>513</td>
<td>2761</td>
</tr>
<tr>
<td>$</td>
<td>S_n(1324)</td>
<td>$</td>
<td>103</td>
<td>513</td>
<td>2762</td>
</tr>
</tbody>
</table>

⇒ NOT all patterns of length 4 are Wilf-equivalent.

In fact, every pattern of length 4 is Wilf-equivalent to one of:

$$2314 \quad 1234 \quad 1324$$

What is known?

- I. Gessel (1990) gave a formula for $|S_n(1234)|$
- M. Bóna (1997) gave a formula for $|S_n(2314)|$

Open Problem

*Find a formula for $|S_n(1324)|$.*
Definition
A **Ferrers Board** $F$ is a square array of boxes with a “bite” taken out of the northeast corner.
Rook Placements

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\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
|   |   |   |   |   |   |   |   |
\hline
|   |   |   |   |   |   |   |   |
\hline
|   |   |   |   |   |   |   |   |
\hline
|   |   |   |   |   |   |   |   |
\hline
|   |   |   |   |   |   |   |   |
\hline
\end{array}
\]
Rook Placements

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A full rook placement (f.r.p.) on $F$ is a placement of markers with EXACTLY one in each row and column.
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A **Ferrers Board** $F$ is a square array of boxes with a “bite” taken out of the northeast corner.

![Diagram of Ferrers Board]

A **full rook placement** (f.r.p.) on $F$ is a placement of markers with *EXACTLY* one in each row and column.
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A Ferrers Board $F$ is a square array of boxes with a “bite” taken out of the northeast corner.

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Notation
$\mathcal{R}_F =$ set of all f.r.p.’s on the fixed board $F$
Rook Placements

This f.r.p. contains the pattern 312

This f.r.p. avoids the pattern 231

Notation

$R_F(\tau) = \text{set of all f.r.p. on } F \text{ that avoid } \tau$. 
Rook Placements

This f.r.p. contains the pattern 312
Rook Placements

This f.r.p. **contains** the pattern 312
This f.r.p. *contains* the pattern 312
Rook Placements

- This f.r.p. **contains** the pattern 312
- This f.r.p. **avoids** the pattern 231
Rook Placements

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Rook Placements

Definition

We say two patterns $\sigma, \tau \in S_k$ are **shape-Wilf-equivalent** and write $\sigma \sim \tau$ if for every Ferrers board $F$

$$|R_F(\sigma)| = |R_F(\tau)|.$$
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Note: shape-Wilf equivalence $\Rightarrow$ Wilf-equivalence.
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  - Complicated proof
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- We give a simple proof that $231 \sim 312$
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Dyck Paths

A Dyck path of size $n$ is a path that:
- starts at the origin
- ends at the point $(2^n, 0)$
- never goes below the x-axis
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- starts at the origin
- ends at the point $(2n, 0)$
A **Dyck path** of size $n$ is a path that:

- starts at the origin
- ends at the point $(2n, 0)$
- never goes below the x-axis
We label the Dyck path so that:

- **Monotonicity**: +1 for an up step and −1 for a down step.
- **Zero Condition**: All zeros lie precisely on the x-axis.
- **Tunnel Property**: "Left" ≤ "Right".
We label the Dyck path so that:

- **Monotonicity**
  - $+1/0$ up step and $-1/0$ down step

- **Zero Condition**
  - All zeros lie precisely on the $x$-axis

- **Tunnel Property**
  - "Left" $\leq$ "Right"
We label the Dyck path so that:

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Labeled Dyck paths

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Our proof of $231 \sim 312$

An outline
Our proof of $231 \sim 312$

An outline

1. 231-avoiding rook placement $\mapsto$ Tunnel property
Our proof of $231 \sim 312$

An outline

1. 231-avoiding rook placement $\leftrightarrow$ Tunnel property
2. Tunnel Property $\leftrightarrow$ Reverse Tunnel Property
Our proof of $231 \sim 312$

An outline

1. 231-avoiding rook placement $\iff$ Tunnel property
2. Tunnel Property $\iff$ Reverse Tunnel Property
3. Reverse Tunnel Property $\iff$ 312-avoiding rook placement
Our proof of $231 \sim 312$

1. 231-avoiding f.r.p. $\Rightarrow$ Tunnel property
Our proof of $231 \sim 312$

1. 231-avoiding f.r.p. $\Rightarrow$ Tunnel property

\[
\begin{array}{cccccccc}
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \\
\times & \times & \times & \\
\times & \\
\end{array}
\]

$\mathcal{R}_F(231)$
Our proof of $231 \sim 312$

1. 231-avoiding f.r.p. $\Rightarrow$ Tunnel property
Our proof of $231 \sim 312$

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$\mathcal{R}_F(231)$
Our proof of $231 \sim 312$

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$$\mathcal{R}_F(231)$$
Our proof of 231 $\sim$ 312

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Our proof of $231 \sim 312$

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$\mathcal{R}_F(231)$
Our proof of $231 \sim 312$

1. $231$-avoiding f.r.p. $\Rightarrow$ Tunnel property
Our proof of $231 \sim 312$

2. Tunnel property $\Rightarrow$ Reverse tunnel property
Our proof of $231 \sim 312$

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2. Tunnel property $\Rightarrow$ Reverse tunnel property
Our proof of $231 \sim 312$

3. Reverse tunnel property $\Rightarrow$ 312-avoiding f.r.p.

\[ \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
2 \\
1 \\
2 \\
1 \\
1 \\
\end{array} \]

\[ \begin{array}{c}
3 \\
2 \\
1 \\
1 \\
1 \\
0 \\
\end{array} \]

\[ \geq \]

Theorem (Bloom–Saracino '11)

This mapping is a bijection between $R_F(231)$ and $R_F(312)$.

$\Rightarrow$ 231 and 312 are shape-Wilf-equivalent.
Our proof of $231 \sim 312$

3. Reverse tunnel property $\Rightarrow$ 312-avoiding f.r.p.

\[
\begin{array}{c}
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 2 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}
\end{array}
\]

$\geq$

$\mathcal{R}_F(312)$
Our proof of $231 \sim 312$

3. Reverse tunnel property $\Rightarrow$ 312-avoiding f.r.p.

Theorem (Bloom–Saracino ’11)

This mapping is a bijection between $\mathcal{R}_F(231)$ and $\mathcal{R}_F(312)$.

$\Rightarrow$ 231 and 312 are shape-Wilf-equivalent.
Generating Functions

The **generating function** for a sequence of integers

\[ a_0, a_1, a_2, a_3, \ldots \]

is the “formal” series

\[ \sum_{n=0}^{\infty} a_n z^n. \]
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▶ “A generating function is a clothesline on which we hang up a sequence of numbers for display” - H. Wilf

Example

Let \( D_n \) be the set of Dyck paths with length \( n \).

\[
C(z) = \sum_{n=0}^{\infty} |D_n| z^n = 1 - \sqrt{1 - 4z^2} = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + \cdots
\]
Generating Functions

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\]
Generating Functions

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Example

Let $\mathcal{D}_n$ be the set of Dyck paths with length $n$.

$$C(z) = \sum_{n=0}^{\infty} |\mathcal{D}_n| z^n = \frac{1 - \sqrt{1 - 4z}}{2z}$$

$$= 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + \cdots$$
In 1990 Bona proved the following celebrated result
\[ \sum_{n=0}^{\infty} |S_n(2314)| z^n = 32z + 20z - 8z^2 - \frac{(1-8z)^3}{2}. \]

Our Proof
Enumerative Results: 2314-Avoiding Permutations

In 1990 Bóna proved the following celebrated result

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\[6257413 \rightarrow S_n(2314)\]
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\[ 6257413 \rightarrow \]

\[ S_n(2314) \quad R_F(231) \]
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\[
6257413 \quad \rightarrow \quad S_n(2314) \quad \rightarrow \quad R_F(231) \quad \leq \quad \text{Graph}
\]
In 2012, D. Callan and V. Kotesovec conjectured that
\[
\sum_{n=0}^{\infty} |S_n(2314, 1234)| z^n = 1 - C(zC(z)) = 1 + z + 2z^2 + 6z^3 + 22z^4 + \cdots
\]
where $C(z)$ is the generating function for the Catalan numbers.

All 231-avoiding f.r.p. are counted by
\[
\frac{54z}{32} - \frac{(1 - 12z)^3}{2} = 1 + z + 3z^2 + 14z^3 + 83z^4 + \cdots
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New enumerative results in the theory of perfect matchings and set partitions.
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Thank you!