Modified Growth Diagrams and the BWX Map ϕ^*

Jonathan Bloom

Dartmouth College

June, 2011

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   $$\sigma_{i_1}\sigma_{i_2} \dots \sigma_{i_k}$$

   and placing in positions $i_1 \dots i_k$ the values

   $$\sigma_{i_2} \dots \sigma_{i_k}\sigma_{i_1},$$

   respectively, and leaving all other entries of σ fixed.

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In other words, ϕ^* is obtained by repeatedly applying the map ϕ until no $k \dots 1$-pattern remains.

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$$\phi^*(\sigma) = \phi^2(\sigma) = 3\ 4\ 1\ 2\ 5$$

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  - This proof is long and difficult.
  - Consequently, in their paper they ask for a better description of the map ϕ^* "on which the commutation theorem would become obvious."

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We answer both Krattenthaler and Bousquet-Mélou and Steingrímsson's questions by providing a reformulation of ϕ^* in terms of growth diagrams that makes the commutation result obvious.

Ferrers Boards & Placements

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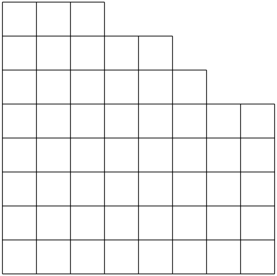

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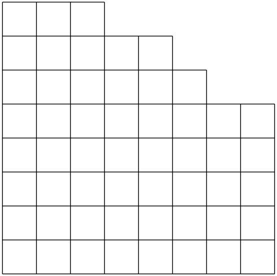

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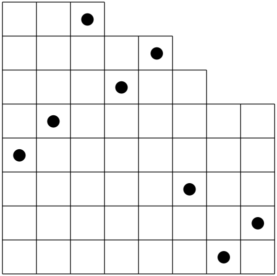

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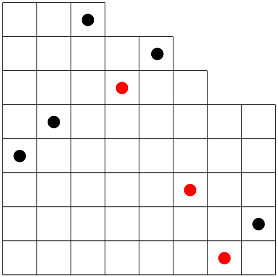

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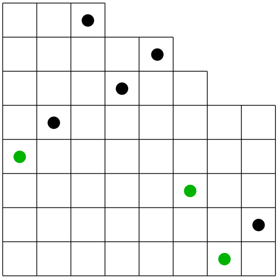

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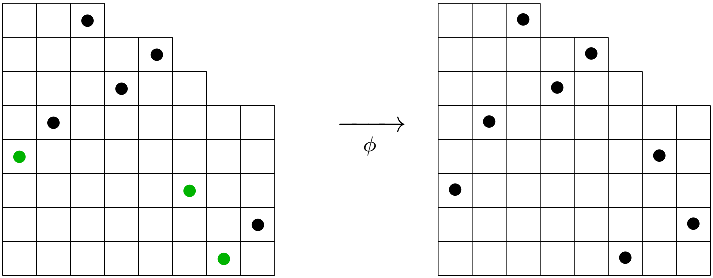

Motivation for the Reformulation of ϕ^*

Consider the Schensted correspondence:

$$\pi = \begin{pmatrix} 1 & 2 & 6 & 7 & 8 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix}$$

$$P = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \cline{1-1} \end{array} \qquad Q = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 6 & 8 \\ \hline 7 & \\ \cline{1-1} \end{array}$$

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Definition: The shape of P and Q is the partition $(\lambda_1 \lambda_2 \ldots \lambda_t)$ such that the top row of P and Q have λ_1 entries, the second row has λ_2 entries, and so on.

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Theorem: The length of the longest decreasing subsequence in π is t.

- Here the subsequence 431 is longest and $t = 3$.

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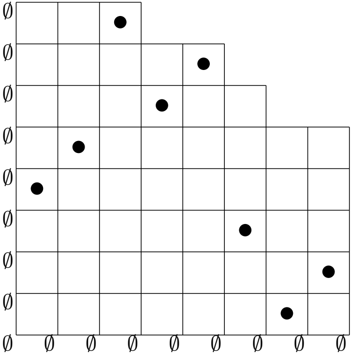

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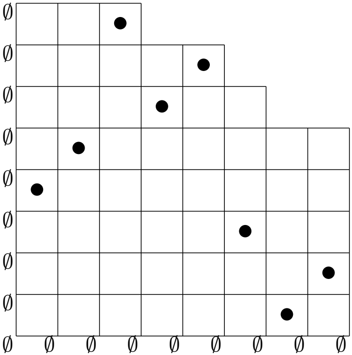
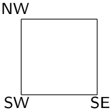


- To determine the partitions on the other corners we use the following rules:

Fomin's Growth Diagram Construction

Given partitions



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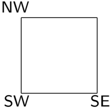
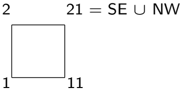


if $SE \neq NW$



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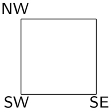

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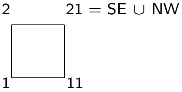

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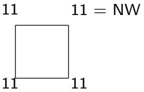

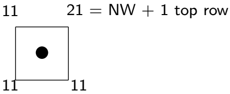

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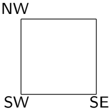

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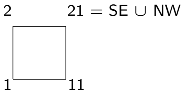

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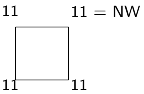

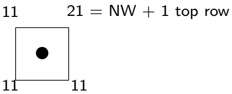

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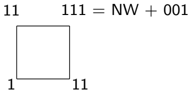

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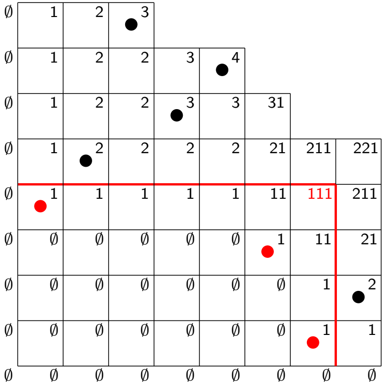
∅ 1 2 3
∅ 1 2 2 3 4
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∅ ∅ ∅ ∅ ∅ ∅ 1 11 21
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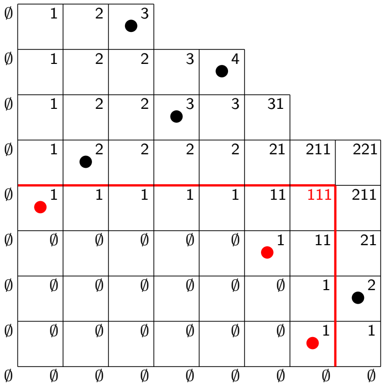

Theorem: Each partition is the shape of the recording/insertion tableaux corresponding to the partial permutation southwest of the partition's location.

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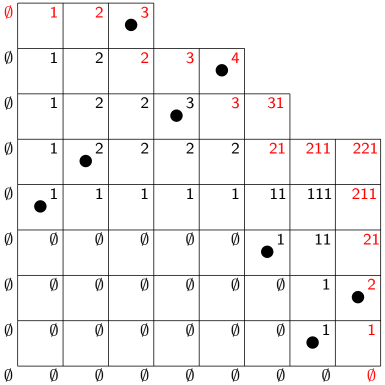

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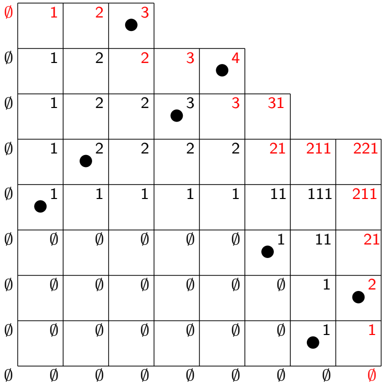

Theorem: Since each step in the Growth Diagram Algorithm (GDA) is reversible then

$$seq(P, F) := (\emptyset, 1, 2, 3, 2, 3, 4, 3, 31, 21, 211, 221, 211, 21, 2, 1, \emptyset)$$

completely determines the placement P.

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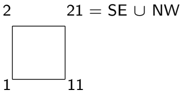

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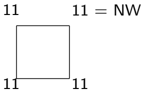

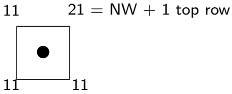

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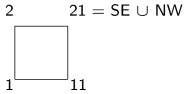

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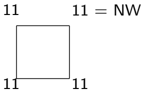

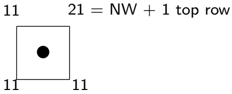

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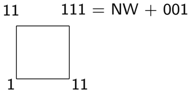
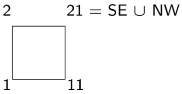


Our Reformulation of ϕ^*

if $SE \neq NW$



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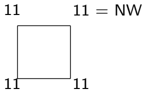

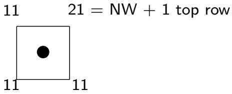

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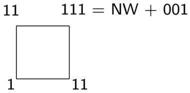
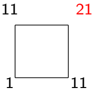


Modified Rule for GDA_k....

*if last rule rule makes $|NE| \geq k$ then



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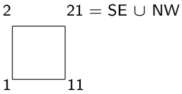

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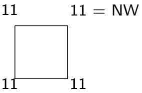

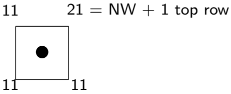

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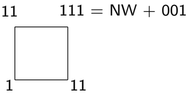

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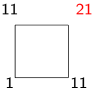

- As the partitions correspond to the shape of the recording/insertion tableaux the modified rule effectively "removes" decreasing subsequence with length $\geq k$ from our placement P.

Our Reformulation of ϕ^*

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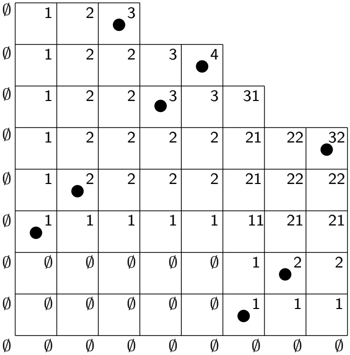
GDA_3 on (P, F)

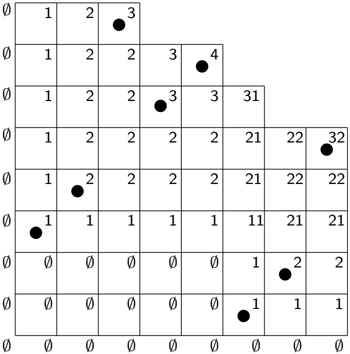
Our Reformulation of ϕ^*

GDA_3 on (P, F) $\qquad$ GDA on $(\phi^*(P), F)$



Our Reformulation of ϕ^*

GDA_3 on (P, F) GDA on $(\phi^*(P), F)$



Main Theorem: For any rook placement P on a Ferrers board F,

$$seq_k(P, F) = seq(\phi^*(P), F)$$

Our Commutation Result

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Definition: For any rook placement P on F, the inverse P' of P is the placement on the conjugate board F' obtained by reflecting F and all the markers for P across the SW-NE diagonal.

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Proof. By the Main Theorem and the note above we have:

$$seq(\phi^*(P'), F) \quad = \quad seq_k(P', F)$$

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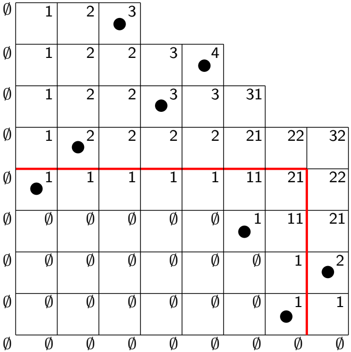
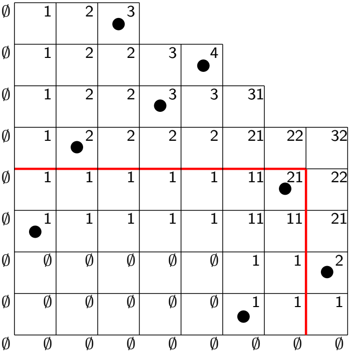
Hence we conclude that $\phi^*(P') = (\phi^*(P))'$.

Idea Behind Main Theorem

GDA_3 on (P, F)



GDA_3 on $(\phi(P), F)$



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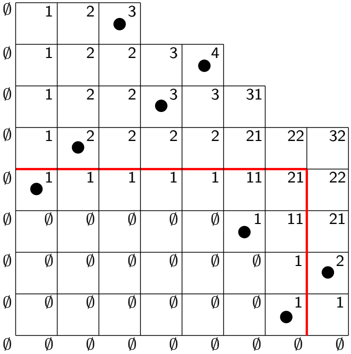

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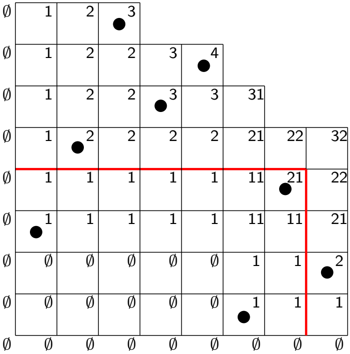

- The red rectangle is the smallest rectangle containing markers moved by ϕ.

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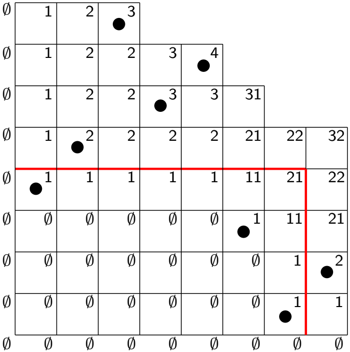

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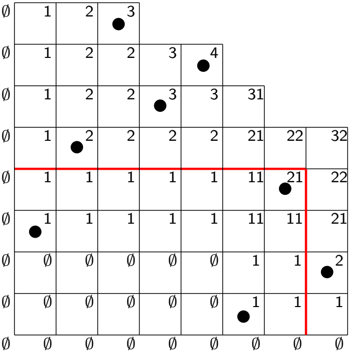

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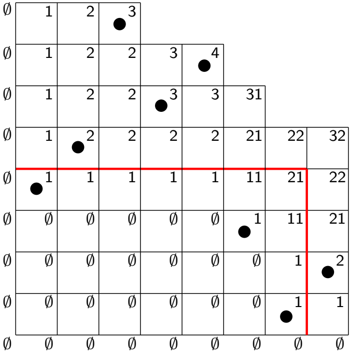

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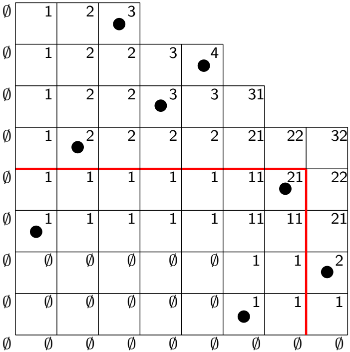

- The red rectangle is the smallest rectangle containing markers moved by ϕ.
- The partitions created by GDA_k along the red are the same in P and $\phi(P)$.
- So GDA_k, outside the red, is identical on P and $\phi(P)$ and we may conclude

$$seq_k(P, F) = seq_k(\phi(P), F)$$

Idea Behind Main Theorem Continued

So...

$$seq_k(P, F) \quad = \quad seq_k(\phi(P), F)$$

Idea Behind Main Theorem Continued

So...

$$\begin{aligned} seq_k(P, F) &= seq_k(\phi(P), F) \\ &= seq_k(\phi^2(P), F) \end{aligned}$$

Idea Behind Main Theorem Continued

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Idea Behind Main Theorem Continued

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Idea Behind Main Theorem Continued

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Yet, $\phi^*(P)$ has no decreasing subsequence of length $\geq k$ hence GDA_k and GDA agree on P. So...

Idea Behind Main Theorem Continued

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$$
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