Another (more refined) look at the Wilf-equivlance of certain length 4 pattern

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Joint Math Meetings - San Antonio 2015

Overview

Part I

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- Another look at Stankova's result that

$$|\operatorname{Av}_n(1423)| = |\operatorname{Av}_n(2413)|$$

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Part II

- Unbalanced-Wilf-equivalence and the Egge triples

$1423 \sim 2413$ revisited

We say two patterns σ and τ are **Wilf-equivalent** provided

$$|\operatorname{Av}_n(\sigma)| = |\operatorname{Av}_n(\tau)|$$

for all n.

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Recall, for patterns of length 4 we have

Class | n	5	6	7	8	9	...
$1423 \sim 2413$	103	512	2740	15485	91245	...
1234	103	513	2761	15767	94359	...
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  - Proof idea: Isomorphic generating trees

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$$\mathrm{Maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i.$$

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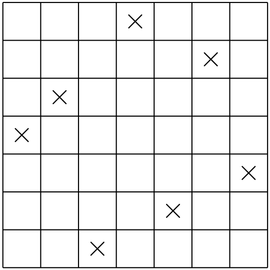

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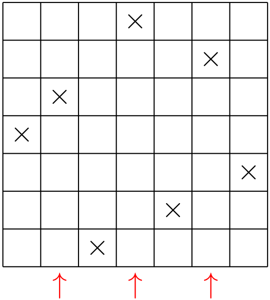

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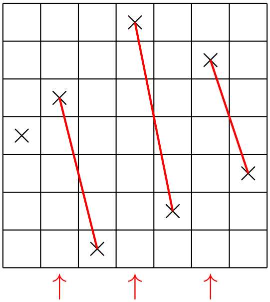

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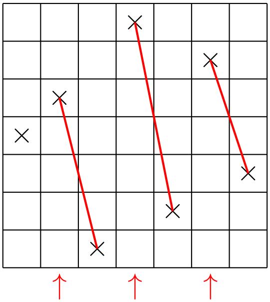

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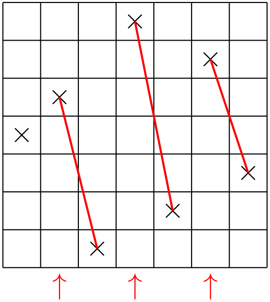

$\mathrm{Des} = \{2, 4, 6\}$
$\mathrm{Maj} = 12$

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provided there is a bijection Θ from $\text{Av}_n(\sigma)$ to $\text{Av}_n(\tau)$ that preserves the Major index,

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$$\sum_{\pi \in \mathsf{Av}(\sigma)} x^{|\pi|} t^{\mathsf{Maj}(\pi)} = \sum_{\pi \in \mathsf{Av}(\tau)} x^{|\pi|} t^{\mathsf{Maj}(\pi)}.$$

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Conjecture (Dokos, et al., 2012)

The patterns 1423 *and* 2413 *are Maj-Wilf-equivalent.*

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Theorem (Bloom, 2014)

There is an explicit bijection

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then $\Theta(\pi) = \pi$.

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- Θ is not the same as Stankova's "implied" bijection.

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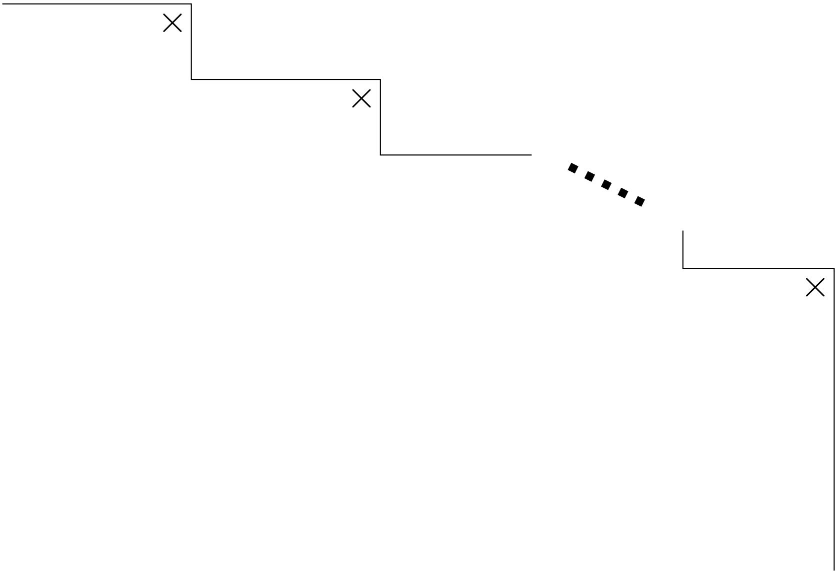
- Θ is not the same as Stankova's "implied" bijection.
- Stankova's isomorphism does not preserve these statistics.

Anatomy of a 1423



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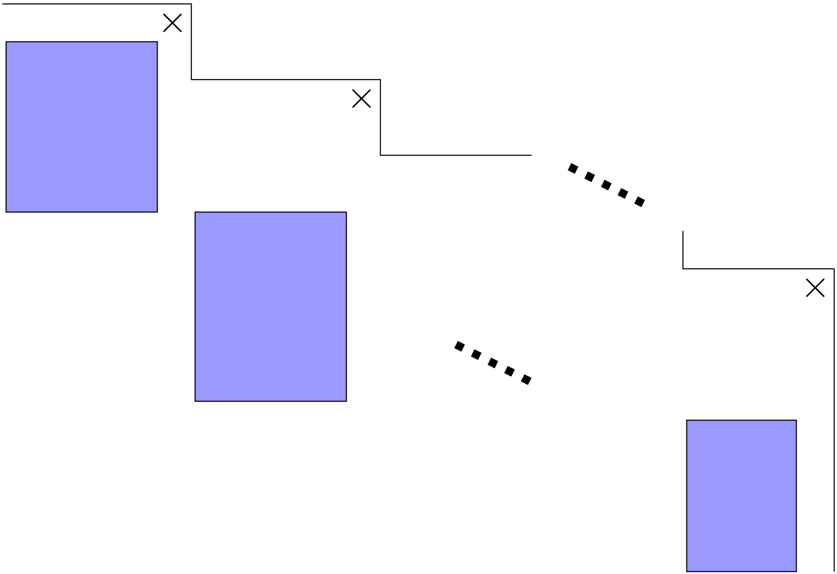

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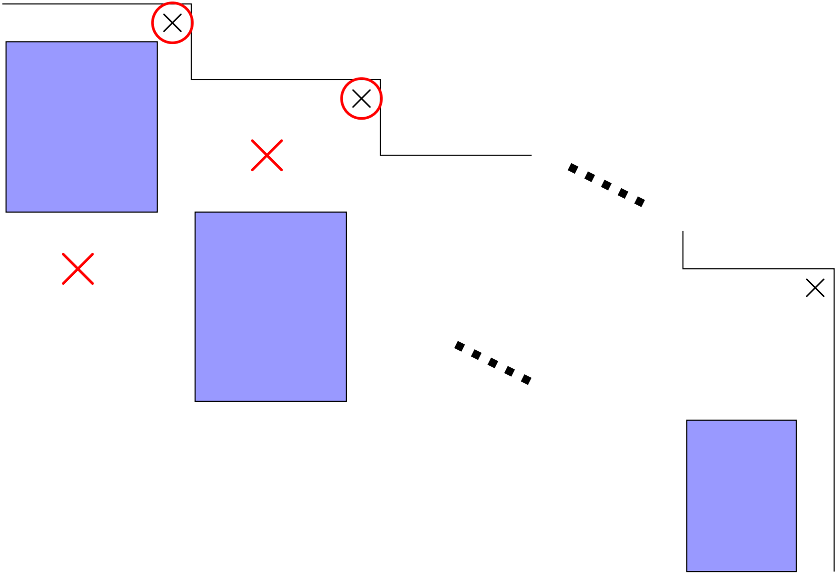
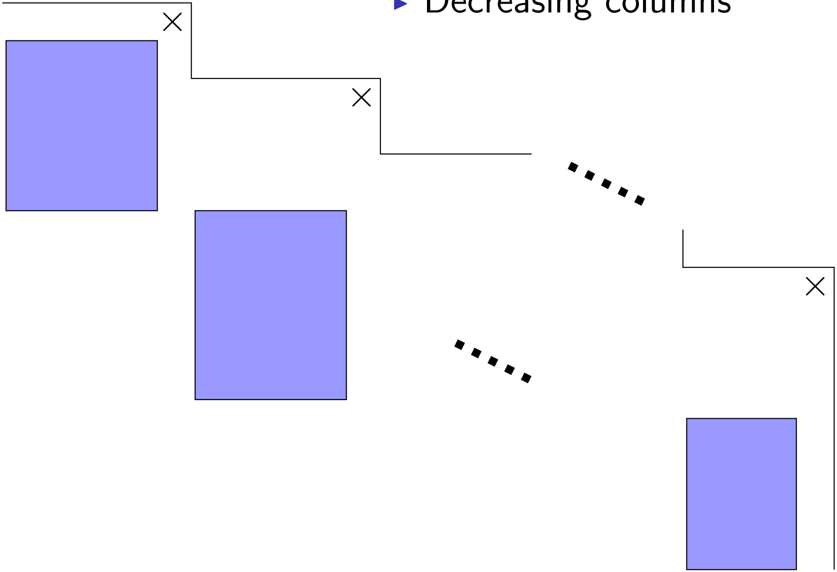


Anatomy of a 1423

- Decreasing columns



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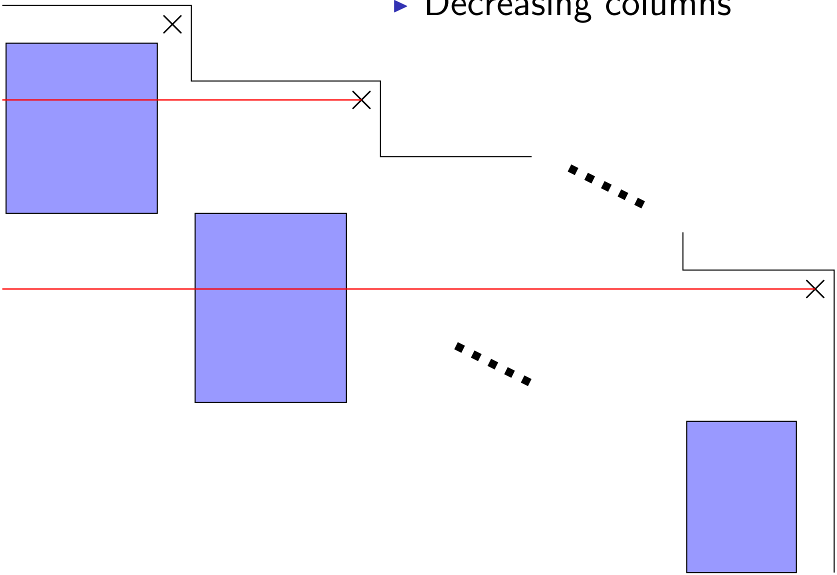

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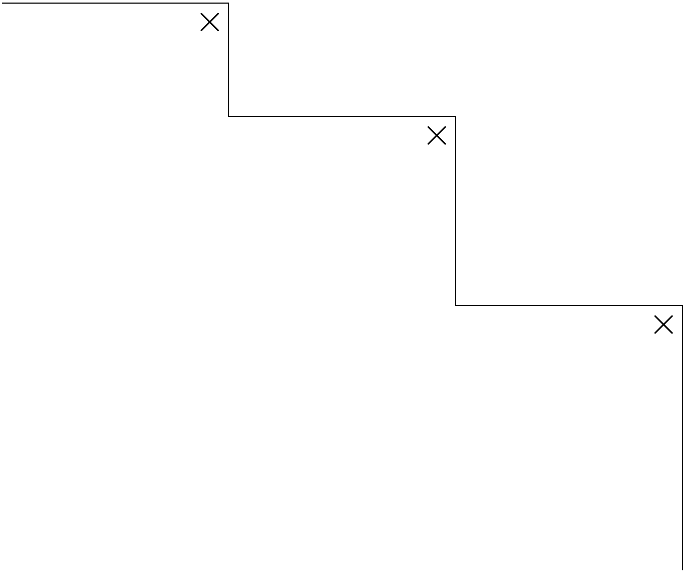
Anatomy of a 2413



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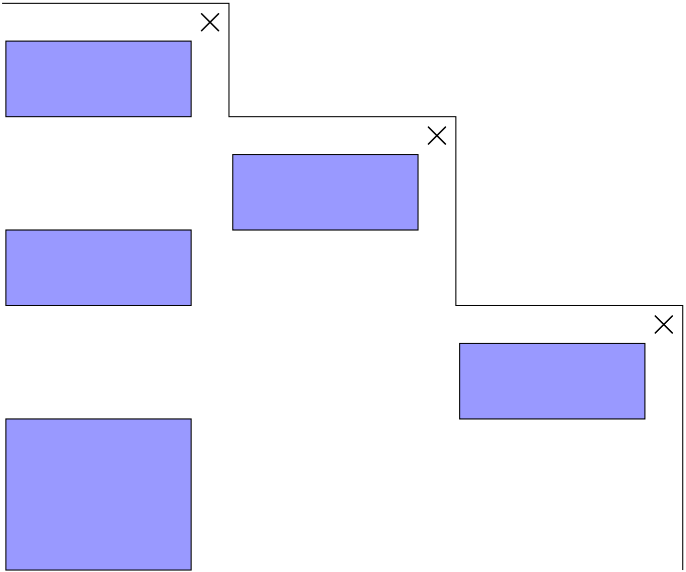

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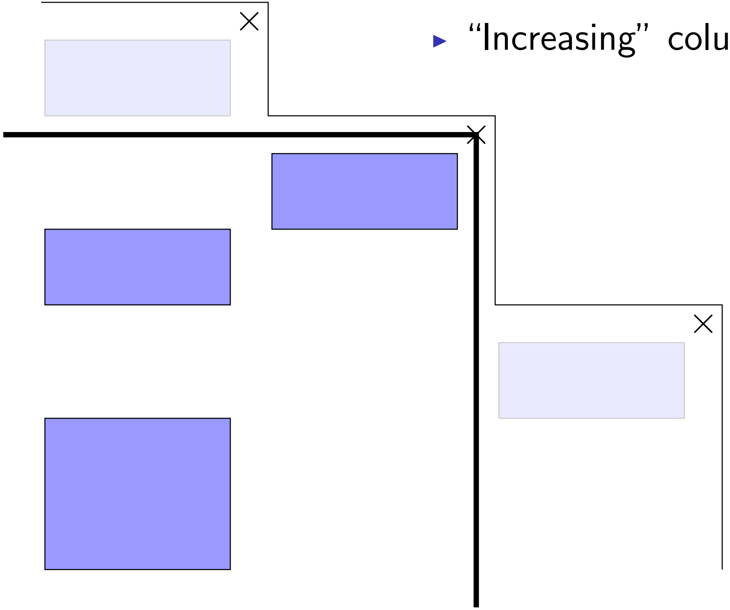

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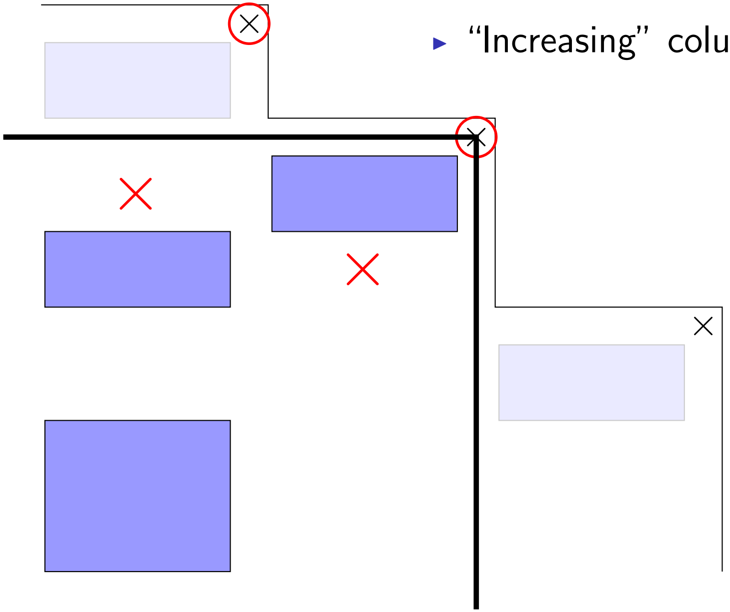

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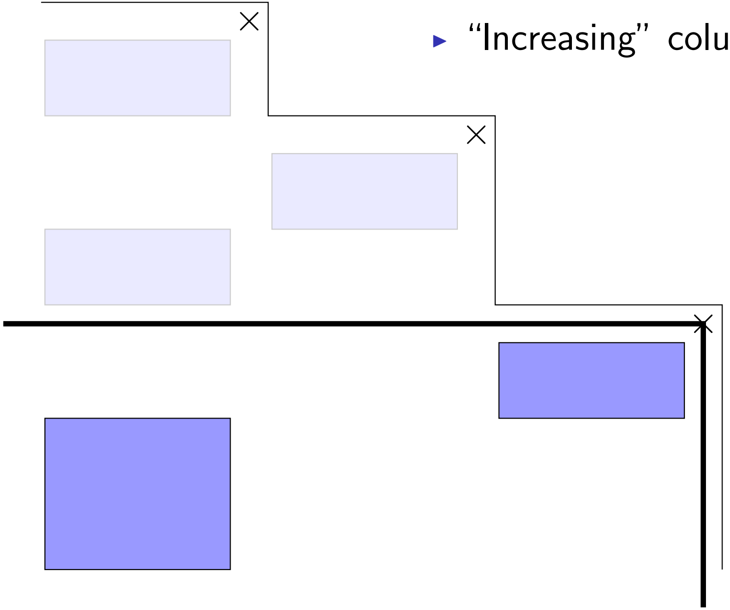

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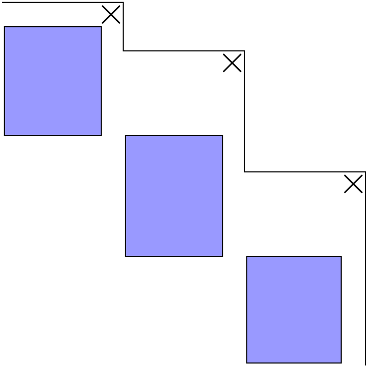
Rough idea behind Θ

Given an arbitrary $\pi \in \mathrm{Av}_n(1423)$:



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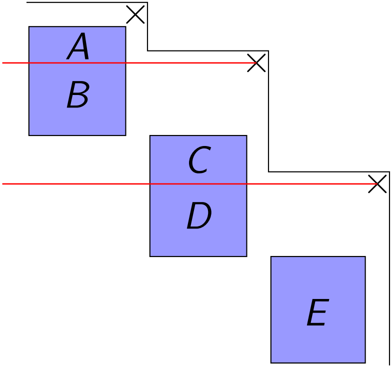

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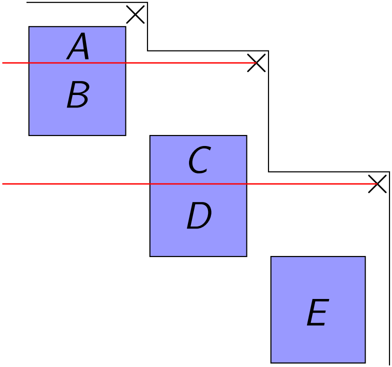

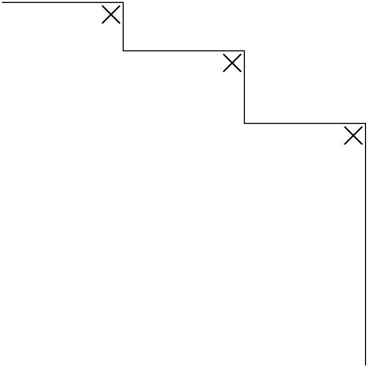

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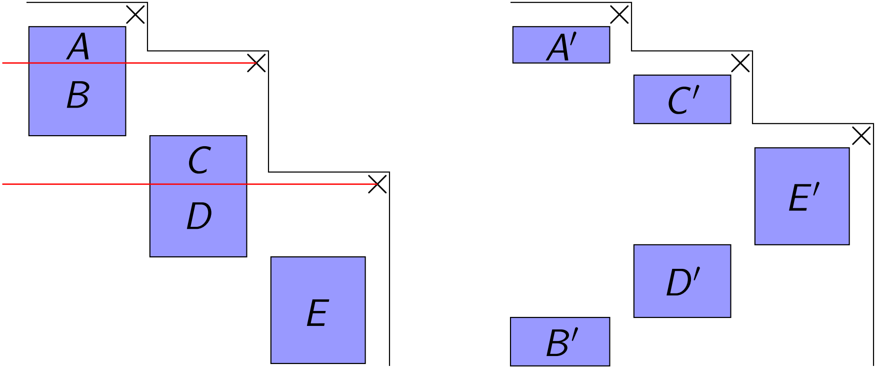

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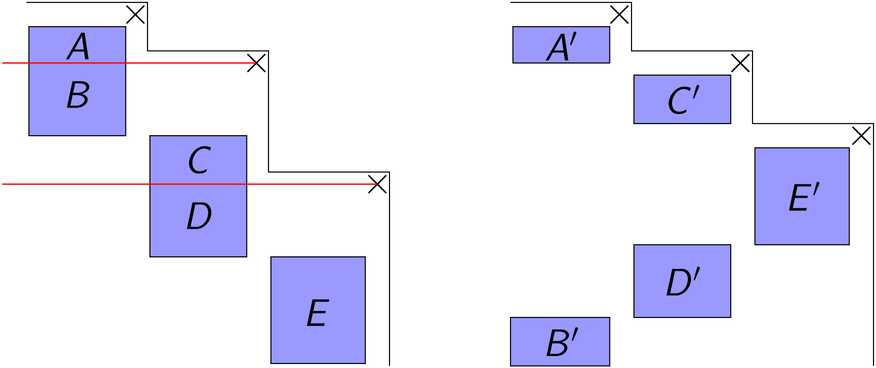

$\pi \in \mathrm{Av}_n(1423)$ $\Theta(\pi)$

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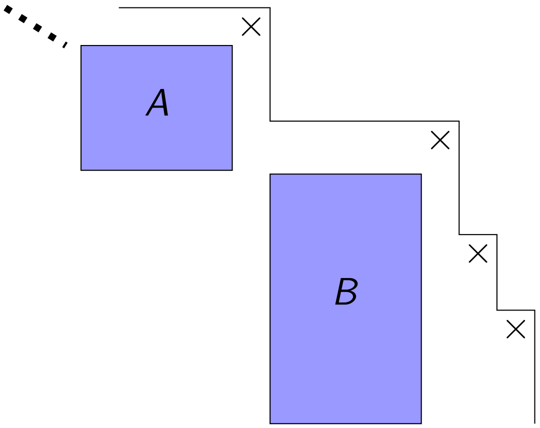

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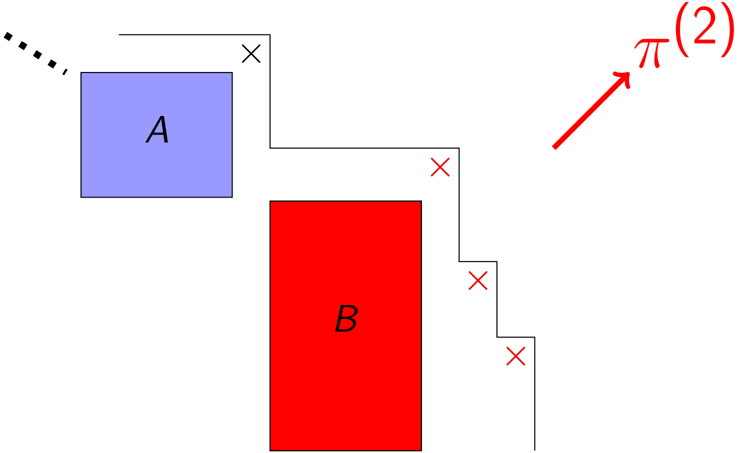

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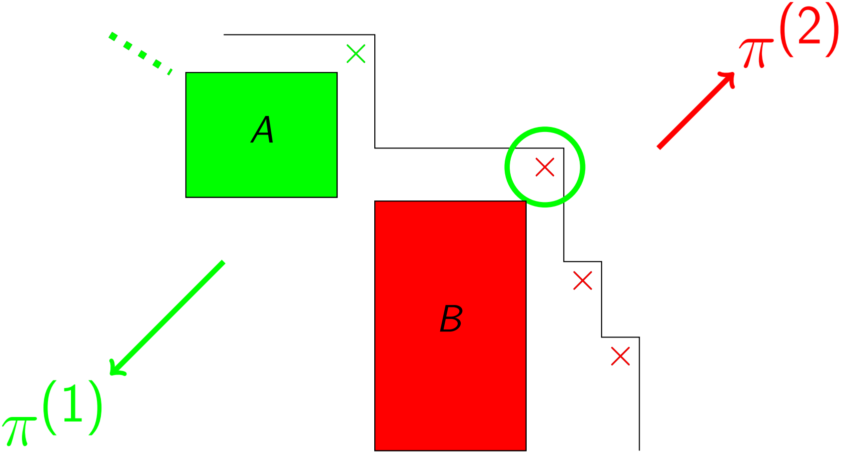

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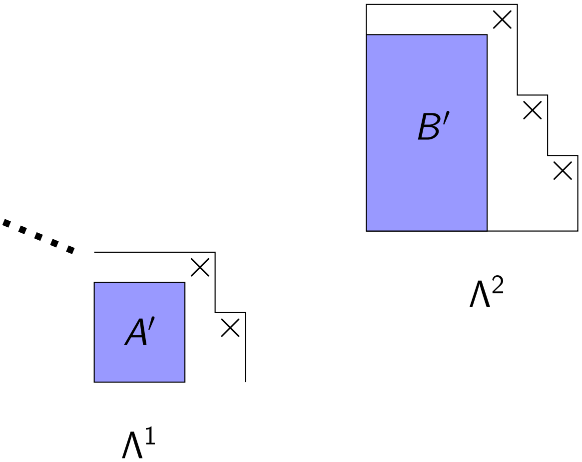

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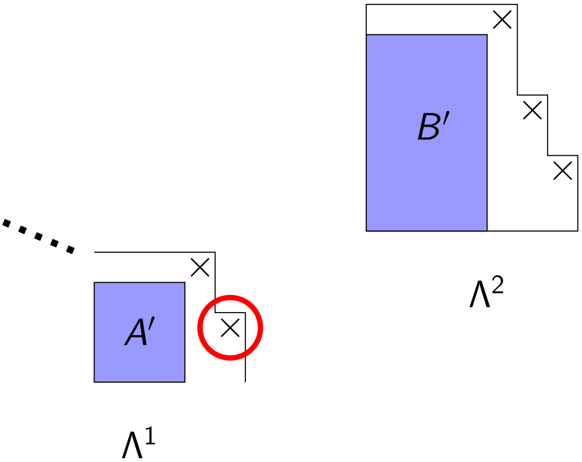

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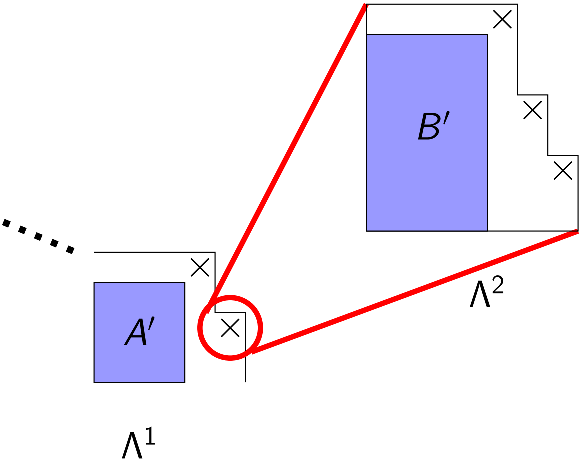

- Lastly, $\Theta(\pi) = \Lambda^1[\Lambda^2]$.

Part II

Egge triples & unbalanced Wilf-equivalences

Conjecture (Egge, AMS Fall Eastern Meeting in 2012)

Fix $\tau \in \{246135, 263514, 254613, 524361, 546132\}$. *Then*

$$\sum_{n\geq 0} |\operatorname{Av}_n(2143, 3142, \tau)| x^n = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2},$$

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  - 263514: simple permutations
  - 254613, 524361, 546132: decomposition using LR-maxima

Unbalanced Wilf-equivalence

Just an example

As separable permutations Av(2413, 3142) are counted by large Schröder numbers, so

$$|\operatorname{Av}_n(2413, 3142)| = |\operatorname{Av}_n(2143, 3142, \tau)|,$$

where $\tau \in \{246135, 263514, 254613, 524361, 546132\}$.

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- Examples of unbalanced Wilf-equivalence abound!

Counting $\mathrm{Av}_n(2143, 3142, 254613)$

If $\pi \in \mathrm{Av}_n(2143, 3142)$, then it looks like:

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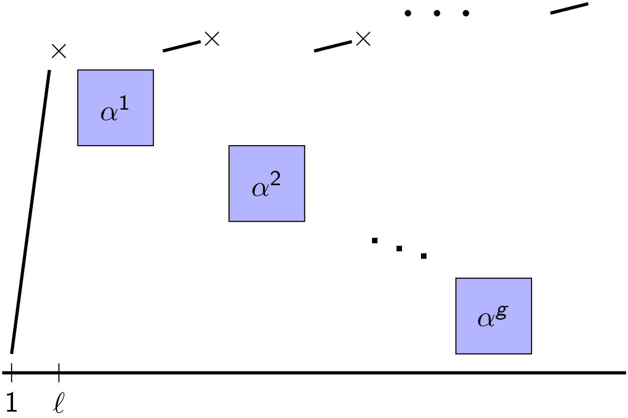

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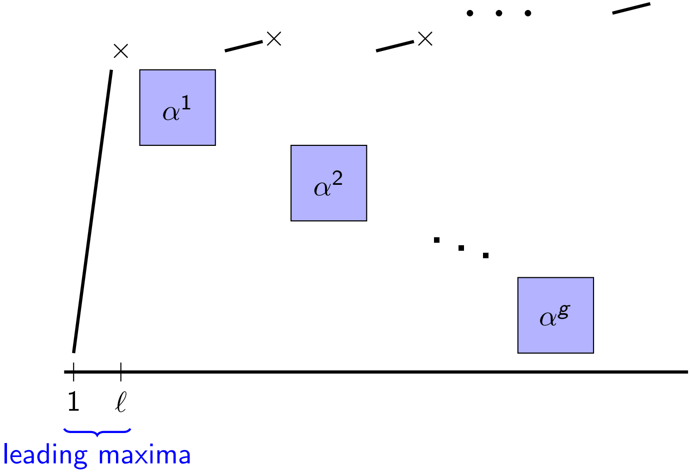

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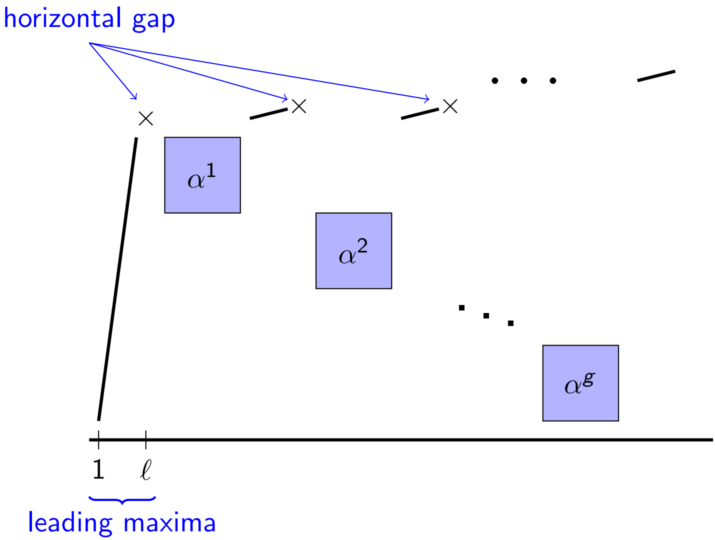

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$$A(x,t) = \sum_{\pi \in \mathsf{Av}(2143,3142,254613)} x^{|\pi|} t^{\#\ \mathrm{lead}(\pi)},$$

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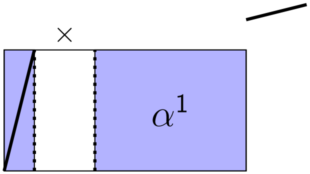
$\pi =$ 

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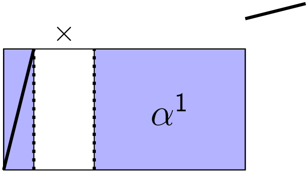

This translates to

$$\frac{txE}{1-x} \quad \text{where} \quad E(x,t) = \frac{B - tA}{1-t} - \frac{1}{1-tx}$$

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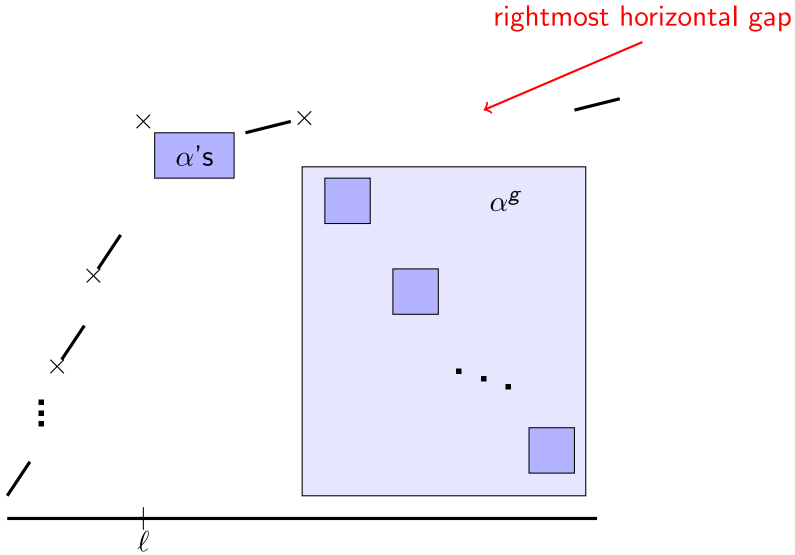
If π has at least 2 horizontal gaps, then



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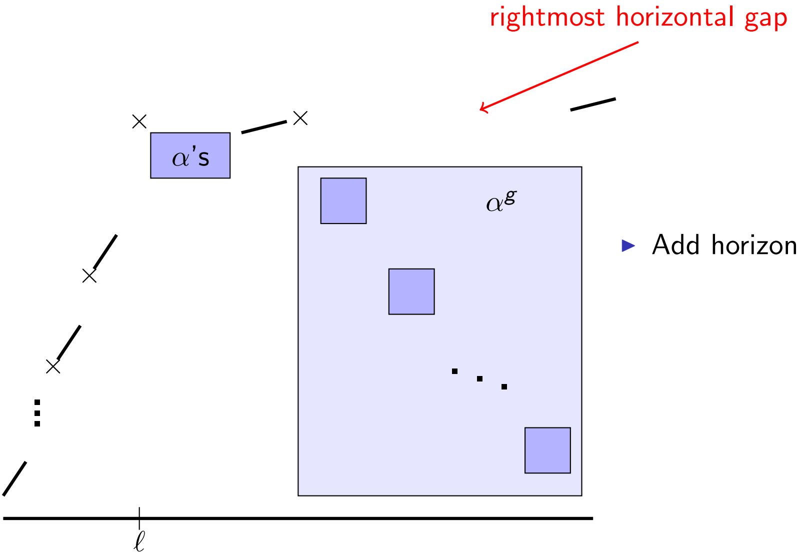

- Add horizontal gap

Counting $\mathrm{Av}_n(2143, 3142, 254613)$

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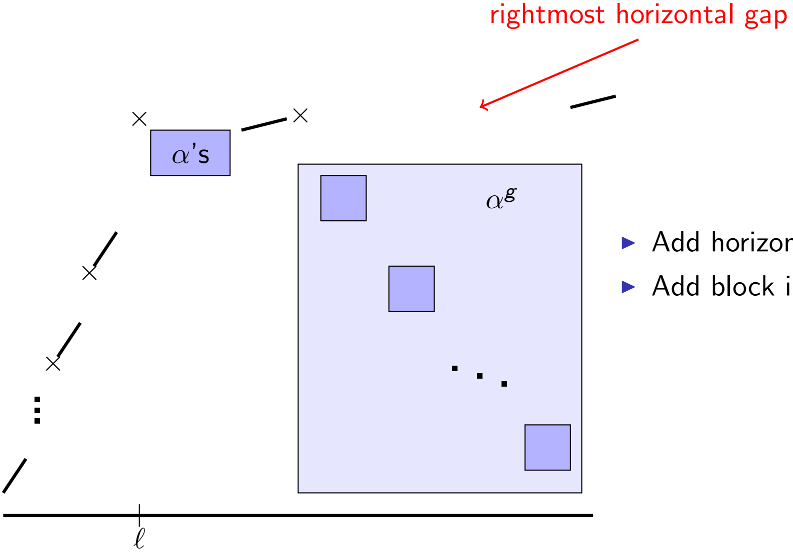

- Add horizontal gap
- Add block into gap

Counting $\text{Av}_n(2143, 3142, 254613)$

Putting this together yields the functional equation

$$A(x,t) = \frac{1}{1-tx} + \frac{txE}{1-x} + \left(A - \frac{1}{1-tx}\right)\left(\frac{x(B-1)}{(1-x)(1-tx)}\right)\left(\frac{1}{1-\frac{tx(B-1)}{1-tx}}\right),$$

where

$$E(x,t) = \frac{B-tA}{1-t} - \frac{1}{1-tx} \qquad \text{and} \qquad B = A(x,1).$$

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where $A_* = A - \dfrac{1}{1-xt}$.

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$$B^3x + B^2x^2 - 3B^2x - B^2 + Bx + 3B - 2$$

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Solving (now) yields

$$\begin{aligned} A(x,1) = B &= \frac{3 - x - \sqrt{1 - 6x + x^2}}{2} \\ &= 1 + x + 2x^2 + 6x^3 + 22x^4 + 90x^5 + \cdots \end{aligned}$$

Thank You!