On two recent conjectures in pattern avoidance

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Rutgers University

Howard University – March 2015
Overview

Part I

▶ On a conjecture of Dokos, et al.
▶ REU group under Sagan
▶ A new statistic-preserving bijection between two old sets

Part II (w/ Burstein)
▶ On a conjecture of Egge (2012)
▶ A collection of pattern classes all counted by the large Schröder numbers
Overview

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Part I

(A statistic-preserving bijection)
Classical Pattern Avoidance

- Let $S_n$ denotes the set of permutations of length $n$
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Example

$$\pi = 7 \ 4 \ 2 \ 6 \ 1 \ 5 \ 3 \in S_7$$
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![Pattern Diagram]

$\pi$ contains the pattern 2 4 1 3 because...

$\pi$ avoids the pattern 1 2 3 because...
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[Diagram of pattern avoidance]
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Patterns
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Patterns

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Wilf-equivalence

Notation

In general, for any \( \sigma \in S_k \) we denote by

\[ \text{Av}_n(\sigma) \]

the set of all permutations (length \( n \)) that avoid \( \sigma \).
Wilf-equivalence

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In general, for any $\sigma \in S_k$ we denote by

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We say two patterns $\sigma, \tau \in S_k$ are **Wilf-equivalent** provided

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|\text{Av}_n(\sigma)| = |\text{Av}_n(\tau)|
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for all $n$. 

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for all $n$. We write $\sigma \sim \tau$. 
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$$|\text{Av}_n(\sigma)| = |\text{Av}_n(\tau)|$$

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All patterns $\tau$ of length 3 are Wilf-equivalent. Moreover,

$$|\text{Av}_n(\tau)| = \frac{1}{1 + n} \binom{2n}{n}.$$
Patterns of length 4

We have:

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<tr>
<th>Class</th>
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Classic results

- There are exactly 3 Wilf-classes in $S_4$
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- We give (first) bijective proof that $1 4 2 3 \sim 2 4 1 3$
- Resolves a conjecture of Dokos, et al. (2012)
Permutation Statistics

Consider the permutation $\pi = 6 \ 5 \ 1 \ 8 \ 2 \ 7 \ 3 \ 4$
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Some statistics:
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- — bonds
Refined Wilf-equivalence

Fix any permutation statistic $f$. 

Conjecture (Dokos, et al., 2012) The patterns 1423 and 2413 are $\text{Maj}$-Wilf-equivalent $\text{Maj}(\pi)$ is sum of descents of $\pi$. 
Refined Wilf-equivalence

Fix any permutation statistic \( f \). We say two patterns \( \sigma, \tau \) are \( f \)-Wilf-equivalent, and write

\[
\sigma \sim_f \tau,
\]

provided there is a bijection \( \Theta : \text{Av}_n(\sigma) \rightarrow \text{Av}_n(\tau) \) that preserves the \( f \) statistic,
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or

$$\sum_{\pi \in \text{Av}(\sigma)} x^{\pi \vert} f(\pi) = \sum_{\pi \in \text{Av}(\tau)} x^{\pi \vert} f(\pi).$$
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$$\sum_{\pi \in \text{Av}(\sigma)} x^{|\pi|} t^{f(\pi)} = \sum_{\pi \in \text{Av}(\tau)} x^{|\pi|} t^{f(\pi)}.$$

Conjecture (Dokos, et al., 2012)

The patterns 1423 and 2413 are Maj-Wilf-equivalent

- Maj($\pi$) is sum of descents of $\pi$. 
1423 ∼ 2413 revisited

Theorem (Bloom, 2014)

There is an explicit bijection

\[ \Theta : \text{Av}_n(1423) \rightarrow \text{Av}_n(2413) \]

such that \( \Theta \) preserves set of descents (hence Major index),

Note ▶ \( \Theta \) is not the same as Stankova’s “implied” bijection.
▶ Stankova’s isomorphism does not preserve these statistics.
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**Note**

- $\Theta$ is not the same as Stankova’s “implied” bijection.
- Stankova’s isomorphism does not preserve these statistics.
Anatomy of a 1423
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- Decreasing columns
Anatomy of a 2413
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- “Increasing” columns
Anatomy of a 2413

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Given $\pi \in \text{Av}_n(1423)$ it decomposes as:
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By induction, \( \Theta : \text{Av}_n \rightarrow \text{Av}_n \) exists and preserves statistics! Including RL maxima! Applying \( \Theta \) to each part maintains structure!
By induction,

\[ \Theta : \text{Av}_n(1423) \rightarrow \text{Av}_n(2413) \]

exists and preserves statistics

- Including RL maxima!
\( \Theta(\pi^{(1)}) = A' \)
\( \Theta(\pi^{(2)}) = B' \)

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exists and preserves statistics
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By induction,

$$\Theta : \text{Av}_n(1423) \to \text{Av}_n(2413)$$

exists and preserves statistics

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⭐ Applying $\Theta$ to each part maintains structure!
Lastly, we must stitch $\Theta(\pi^{(1)})$ and $\Theta(\pi^{(2)})$ back together...
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Doing this we obtain our final result:

\[ \Theta(\pi) = A' \times B' \]
Part II

(Pattern classes & large Schröder numbers)
Large Schröder numbers

The large Schröder are

1, 2, 6, 22, 90, 394, 1806, ... .

They count LOTS!

1. Lattice paths from (0, 0) to (2n, 0) that consist of up/down/over steps – must remain above x-axis.

2. Separable permutations: All permutations built by

where \( \pi \) and \( \sigma \) are separable.
Egge’s motivation

Consider the following table

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**Question:**
Are there any patterns $\tau \in S_6$ such that the sets

$$| \text{Av}_n(2143, 3142, \tau) |$$

are counted by the large Schröder numbers?
Egge triples & unbalanced Wilf-equivalences

Conjecture (Egge, AMS Fall Eastern Meeting in 2012)

Fix $\tau \in \{246135, 254613, 524361, 546132, 263514\}$. Then

$$\sum_{n \geq 0} |\text{Av}_n(2143, 3142, \tau)| x^n = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2},$$

▶ $\text{Av}_n(2143, 3142, \tau)$ is counted by the large Schröder numbers

▶ These values of $\tau$ (and $180^{\circ}$ rotations) are only patterns

Proved...

▶ Burstein and Pantone proved $\tau = 246135$

▶ Bloom and Burstein proved the remaining 4 cases

▶ $263514$: simple permutations

▶ $254613, 524361, 546132$: decomposition using LR-maxima
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Unbalanced Wilf-equivalence

It is well known that the separable permutations, i.e., \( \text{Av}(2413, 3142) \) are also counted by large Schröder numbers, so

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|\text{Av}_n(2413, 3142)| = |\text{Av}_n(2143, 3142, \tau)|,
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where \( \tau \in \{246135, 263514, 254613, 524361, 546132\} \).

▶ Examples of unbalanced Wilf-equivalence abound!
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General phenomenon
Unbalanced Wilf-equivalence

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Let $X$ and $Y$ be two sets of patterns so that for some $k$

$$|X \cap S_k| \neq |Y \cap S_k|.$$
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then, we say \( X \) and \( Y \) are an **unbalanced Wilf-equivalence**.
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Anatomy of (2143, 3142)-avoiders

If $\pi \in Av_n(2143, 3142)$, then it looks like:
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If $\pi \in \text{Av}_n(2143, 3142)$, then it looks like:

\[ \alpha^1 \times \alpha^2 \times \alpha^g \]

leading maxima
Anatomy of $(2143, 3142)$-avoiders

If $\pi \in \text{Av}_n(2143, 3142)$, then it looks like:

- horizontal gap
- leading maxima

\[ \alpha^1 \]
\[ \alpha^2 \]
\[ \alpha^g \]
Counting $\tau = 254613$

Idea We consider three cases:

- No horizontal gaps
- Exactly 1 horizontal gap
- At least 2 horizontal gaps

Set

$$A(t, x) = \sum_{\pi \in \text{Av}(2143, 3142, \tau)} x^{\left| \pi \right|} t^{\ell(\pi)},$$

where $\ell(\pi)$ is the number of leading maxima in $\pi$. 
Case 1: No Horizontal gap

\[ \pi \in Av_n(2143, 3142, \tau) \text{ has no horizontal gap iff } \]

\[ \pi = 1 \ 2 \ \ldots \ \ n. \]
Counting \( \tau = 254613 \)

**Case 1: No Horizontal gap**

\( \pi \in \text{Av}_n(2143, 3142, \tau) \) has no horizontal gap iff

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Counted by

\[ \frac{1}{1 - tx}. \]
Counting $\tau = 254613$

**Case 2:** Exactly 1 horizontal gap

This translates to $t x E_1 - x$ where $E(t, x) = B - t A_1 - t - 1 - t x$ and $B = A(1, x)$. 
Counting $\tau = 254613$

**Case 2: Exactly 1 horizontal gap**

This translates to $t x E_1 - x$ where $E(t, x) = B - t A_1 - t_1 - x$ and $B = A(1, x)$. 
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**Case 2:** Exactly 1 horizontal gap

This translates to $t x E_1 - x$ where $E(t, x) = B - tA_1 - t - 1$ and $B = A(1, x)$. 

\[ \alpha^1 \]
Counting $\tau = 254613$

Case 2: Exactly 1 horizontal gap

This translates to $t_x E_1 - x$ where $E(t, x) = B - t A_1 - t - 1 - t x$ and $B = A_1(1, x)$. 
Counting $\tau = 254613$

Case 2: Exactly 1 horizontal gap

This translates to $t \times E - x$ where $E(t, x) = B - tA - 1 - t - x$ and $B = A(1, x)$. 
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Case 2: Exactly 1 horizontal gap

This translates to

$$\frac{txE}{1 - x}$$

where

$$E(t, x) = \frac{B - tA}{1 - t} - \frac{1}{1 - tx}$$

and

$$B = A(1, x).$$
Counting $\tau = 254613$

**Case 3:** At least 2 horizontal gap

\[ \alpha^g \]

rightmost horizontal gap

$\alpha$'s
Counting $\tau = 254613$

**Case 3:** At least 2 horizontal gap

- Add horizontal gap
- Add block into gap
Counting $\tau = 254613$

Case 3: At least 2 horizontal gap

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All Together...

$$A(t, x) = \frac{1}{1 - tx} + \frac{txE}{1 - x}$$

$$+ \left( A - \frac{1}{1 - tx} \right) \left( \frac{x(B - 1)}{(1 - x)(1 - tx)} \right) \left( \frac{1}{1 - \frac{tx(B - 1)}{1 - tx}} \right),$$

where

$$E(t, x) = \frac{B - tA}{1 - t} - \frac{1}{1 - tx} \quad \text{and} \quad B = A(1, x).$$
Counting $\tau = 254613$

With a bit of algebra (thanks to Mathematica)
Counting $\tau = 254613$

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\[
\left( \frac{Bt^3x^2 + Bt^2x^2 - Bt^2x - Btx^2 + Bx - t^2x + t - 1}{(1 - t)(1 - x)(1 - Btx)} \right) A_*
\]

\[
= \frac{xt}{1 - x} \left( \frac{Btx - B + 1}{(t - 1)(tx - 1)} \right)
\]

where $A_* = A - \frac{1}{1 - xt}$. 

▶ Setting the kernel to zero

\[
0 = Bt^3x^2 + Bt^2x^2 - Bt^2x - Btx^2 + Bx - t^2x + t - 1.
\]

▶ Directly solving fails

▶ Let $t = t(x)$ be the desired solution

▶ The RHS yields:

\[
Btx - B + 1
\]

\[
= (t - 1)(tx - 1)
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Counting $\tau = 254613$

With a bit of algebra (thanks to Mathematica)

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- Directly solving fails
- Let $t = t(x)$ be the desired solution
  - The RHS yields: $Bxt(x) = B - 1$
Counting $\tau = 254613$

Using the fact that $B_{xt}(x) = B - 1$, the kernel becomes

\begin{align*}
B_3 x &+ B_2 x^2 - 3 B_2 x - B_2 + B x + 3 B - 2 \\
&= (xB - 1)(B_2 + (x - 3)B + 2) \\
&= 1 + x + 2x^2 + 6x^3 + 22x^4 + 90x^5 + \cdots
\end{align*}
Counting $\tau = 254613$

Using the fact that $Bxt(x) = B - 1$, the kernel becomes

$$B^3x + B^2x^2 - 3B^2x - B^2 + Bx + 3B - 2$$
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Using the fact that $\mathcal{B}xt(x) = B - 1$, the kernel becomes

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Using the fact that $B^x t(x) = B − 1$, the kernel becomes


Solving (now) yields

$$A(1, x) = B = \frac{3 − x − \sqrt{1 − 6x + x^2}}{2} = 1 + x + 2x^2 + 6x^3 + 22x^4 + 90x^5 + \cdots$$
Thank You!