Pattern Avoidance in Ferrers Boards and Set Partitions

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February, 2013
Ferrers Boards & Full Rook Placements

A Ferrers Board $F$ is an $n \times n$ array of unit squares with a “bite” taken out of the N.E. section.

**Definition**

A full rook placement (f.r.p.) on $F$ is a way of placing $n$ rooks in $F$ such that no two are in the same row or column.

**Notation**

- $F_n$ is the set of all Ferrers boards that admit a f.r.p. of $n$ rooks.
- $R_F$ is the set of all f.r.p. on $F \in F_n$ and $R_n := \bigcup F \in F_n R_F$. 
A Ferrers Board $F$ is an $n \times n$ array of unit squares with a “bite” taken out of the N.E. section.

For example

```
+ + +  +  +  +  +  +  +  
+  +  +  +  +  +  +  +  +  
+  +  +  +  +  +  +  +  +  
+  +  +  +  +  +  +  +  +  
```

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- $R_n := \bigcup_{F \in F_n} R_F$. 
Ferrers Boards & Full Rook Placements

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For example

\[ \begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array} \]
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\[ D_F \]
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![Ferrers Board Diagram]

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For example

```
D_F

×
×
×
×
```

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- $\mathcal{F}_n$ is the set of all Ferrers boards that admit a f.r.p. of $n$ rooks.
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![Diagram of a Ferrers Board]

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\mathcal{R}_n := \bigcup_{F \in \mathcal{F}_n} \mathcal{R}_F.
\]
Pattern Avoidance in Rook Placements

Definition
We say a f.r.p. on $F \in \mathcal{F}_n$ avoids some pattern $\sigma \in S_k$ if the following happens. For any rectangle inside $F$ the “permutation” in this rectangle avoids $\sigma$ in the classical sense.
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For example,

```
  × × ×
  ×   ×
  ×   ×
  ×   ×
  ×   ×
```

avoids 132 but does NOT avoid 312. (Important: Read using cartesian coordinates!!)
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For example,

\begin{center}
\begin{tikzpicture}
\draw (0,0) grid (5,5);
\draw[red] (1,2) -- (3,4);
\draw[red] (2,3) -- (4,5);
\draw[red] (3,4) -- (4,5);
\end{tikzpicture}
\end{center}

avoids 132 but does NOT avoid 312. (Important: Read using cartesian coordinates!!)
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For example, avoids 132 but does NOT avoid 312. (Important: Read using cartesian coordinates!!)

Notation

- $\mathcal{R}_F(\sigma)$ is the set of all rook placements on $F \in \mathcal{F}_n$ that avoid $\sigma$ and

$$\mathcal{R}_n(\sigma) := \bigcup_{F \in \mathcal{F}_n} \mathcal{R}_F(\sigma).$$
Definition
We say two patterns $\sigma, \tau \in S_k$ are shape-Wilf-equivalent if for any Ferrers board $F \in \mathcal{F}_n$

$$|\mathcal{R}_F(\sigma)| = |\mathcal{R}_F(\tau)|.$$ 

In this case we write $\sigma \sim \tau$. 
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History (k=3)

- $231 \sim 312$,  $123 \sim 321 \sim 213$,  $132$
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- $|\mathcal{R}_n(231)| \leq |\mathcal{R}_n(123)| \leq |\mathcal{R}_n(132)|$
  - Z. Stankova
What is Still Needed....

(Open) Problems:

1 Simple proof that $|\mathcal{R}_n(231)| = |D_n^2|$. 

- M. Bousquet-Mélou and G. Xin did the "123" case.
We will show 1, 2 & the "231" case of 4. In addition, we will use:

- Give a new proof counting $|S_n(2314)| = |S_n(1342)|$.
- Analyze the shape-Wilf-equivalence for pairs of patterns with length 3.
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1. Simple proof that $|\mathcal{R}_n(231)| = |\mathcal{D}_n^2|$.
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3. Enumeration of $R_n(132)$.
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(Open) Problems:

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2. Enumeration of $R_n(231)$.
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4. Enumeration of the “231” and “132” classes for set partition.

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1 Simple proof that \(|R_n(231)| = |D^2_n|\).
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A Simple Proof that $|R_n(213)| = |D_n^2|$. 

Notation

Let $D_n$ be the set of Dyck paths with semilength $n$. 

Proof by example:
A Simple Proof that $|\mathcal{R}_n(213)| = |\mathcal{D}_n^2|$.

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Let $\mathcal{D}_n$ be the set of Dyck paths with semilength $n$.

Lemma

$F \in \mathcal{F}_n$ iff $D_F \in \mathcal{D}_n$. Therefore $\mathcal{F}_n$ and $\mathcal{D}_n$ are in bijection!
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Proof by example:

Dyck Path!!
A Simple Proof that $|\mathcal{R}_n(213)| = |\mathcal{D}^2_n|$.

Theorem (J. Bloom, S. Elizalde)

There exists an explicit (and painfully simple!) bijection

$$\mathcal{R}_n(213) \rightarrow \mathcal{D}^2_n.$$
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Theorem (J. Bloom, S. Elizalde)

There exists an explicit (and painfully simple!) bijection

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Observe that $54132 \in \mathcal{S}(213)$ results from the standard bijection $\mathcal{S}(213) \rightarrow \mathcal{D}_n^2$. 

\begin{tikzpicture}
  
  \begin{scope}[every node/.style={draw,minimum size=5mm}]
    \draw (-.5,-.5) grid (4.5,4.5);
    \node (a) at (0,0) {$\times$};
    \node (b) at (1,1) {$\times$};
    \node (c) at (2,2) {$\times$};
    \node (d) at (3,3) {$\times$};
    \node (e) at (4,4) {$\times$};
  \end{scope}
\end{tikzpicture}
A Simple Proof that $|\mathcal{R}_n(213)| = |\mathcal{D}^2_n|$.

Theorem (J. Bloom, S. Elizalde)

*There exists an explicit (and painfully simple!) bijection*

$$\mathcal{R}_n(213) \rightarrow \mathcal{D}^2_n.$$

![Diagram showing the bijection between $\mathcal{R}_n(213)$ and $\mathcal{D}^2_n$.](image)
A Simple Proof that $|\mathcal{R}_n(213)| = |\mathcal{D}^2_n|$.

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There exists an explicit (and painfully simple!) bijection

$$\mathcal{R}_n(213) \rightarrow \mathcal{D}_n^2.$$ 

- Observe that $54132 \in S(213)$
  - Blue path results from the standard bijection $S(213) \rightarrow \mathcal{D}_n$. 
The Enumeratation of $R_n(231)$

Theorem (J. Bloom, D. Saracino)

Let $F \in F_n$. There exists an (simple) explicit bijection

$$\Pi : R_F(231) \rightarrow L_F,$$

where $L_F$ is a set of “special” labelings of the border of $F$. 
The Enumeratation of $\mathcal{R}_n(231)$

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An example:
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An example:

```
  0 1 1 1
  X
  X
  X
  X
  X
  X
  X
```
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An example:

Critical Properties:

• Monotone Property - Increase $\leq 1$ over east step.
• Decrease $\leq 1$ over south step.
• Zero Condition - Zero labels are those on the diagonal.
• Diagonal Property - For any diagonal: “Top” $\leq$ “Bottom”.

Diagram:

```
0 1 1 1
× 1 2
   × 2
   ×× 1
   ×× 1
   2
   2
   1
   1
   2
   0
```
The Enumeratation of $\mathcal{R}_n(231)$

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- **Diagonal Property**
  - For any diagonal: “Top” $\leq$ “Bottom”.
Theorem (J. Bloom, S. Elizalde)

\[
\sum_{n \geq 0} |\mathcal{R}_n(231)| z^n = \sum_{n \geq 0} |\mathcal{L}_n| z^n = \frac{54z}{1 + 36z - (1 - 12z)^{3/2}},
\]

where \(\mathcal{L}_n \coloneqq \bigcup_{F \in \mathcal{F}_n} \mathcal{L}_F\). Further, we obtain

\[
|\mathcal{R}_n(231)| \sim \frac{3^3}{2^5 \sqrt{\pi n^5}} 12^n.
\]
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\sum_{n \geq 0} |R_n(231)| z^n = \sum_{n \geq 0} |L_n| z^n = \frac{54z}{1 + 36z - (1 - 12z)^{3/2}},
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Idea behind the proof:

- Tweaking the standard decompositions for Dyck paths we get a functional equation.
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- Tweaking the standard decompositions for Dyck paths we get a functional equation.
- We solve this functional equation using the quadratic method developed by Tutte for counting rooted planar maps.
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\sum_{n \geq 0} |R_n(231)|z^n = \sum_{n \geq 0} |L_n|z^n = \frac{54z}{1 + 36z - (1 - 12z)^{3/2}},
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Idea behind the proof:

- Tweaking the standard decompositions for Dyck paths we get a functional equation.
- We solve this functional equation using the \textit{quadratic method} developed by Tutte for counting rooted planar maps.
- Interestingly, labelings that have only the Monotone Property and the Diagonal Property are in bijection with rooted planar maps!!
The pattern 2314

Any $\pi \in S_n(2314)$ may be thought of as a full rook placement on a minimal Ferrers board.
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Any $\pi \in S_n(2314)$ may be thought of as a full rook placement on a minimal Ferrers board.

For example, $\pi = 7165324 \in S_7(2314)$ maps to

\[
\begin{array}{cccccccc}
\times & & & & & & & \\
& \times & & & & & & \\
& & \times & & & & & \\
& & & \times & & & & \\
& & & & \times & & & \\
& & & & & \times & & \\
& & & & & & \times & \\
& & & & & & & \times
\end{array}
\]
The pattern 2314

Any $\pi \in S_n(2314)$ may be thought of as a full rook placement on a **minimal** Ferrers board.

For example, $\pi = 7165324 \in S_7(2314)$ maps to

- This f.r.p. is in $\mathcal{R}_7(231)$. 
The pattern 2314

Any $\pi \in S_n(2314)$ may be thought of as a full rook placement on a minimal Ferrers board.

For example, $\pi = 7165324 \in S_7(2314)$ maps to

```
      0 1
     × 0 1 2
    × × 1 2
   × × × 1 2
  × × × × 2
 × × × × × 2
× × × × × × 1
× × × × × × × 0
```

• This f.r.p. is in $\mathcal{R}_7(231)$. 
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  ×  2  2
  ×     2
  ×     1
  ×     0
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- The labels rounding any peak are always $a, a+1, a$. Call this the peak property.
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```
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  ×
  ×
  ×
  ×
```

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- The labels rounding any peak are always $a, a+1, a$. Call this the peak property.

Notation

- Let $\mathcal{L}_n^\times \subset \mathcal{L}_n$ be all labelings with the peak property.
The pattern 2314

Lemma (J. Bloom, S. Elizalde)

This composition of maps gives a bijection

\[ S_n(2314) \rightarrow \mathcal{L}_n^\times(231), \]

and therefore

\[ \sum_{n \geq 0} |S_n(2314)|z^n = \sum_{n \geq 0} |\mathcal{L}_n^\times|z^n. \]
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- To count \( \mathcal{L}_n^\times \) is simply a matter of “tweaking” the method used to count \( \mathcal{L}_n \).
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• To count \( L_n^\times \) is simply a matter of “tweaking” the method used to count \( L_n \).

Doing so we obtain

\[ \sum_{n \geq 0} |S_n(2314)|z^n = \frac{32z}{1 + 20z - 8z^2 - (1 - 8z)^{3/2}}. \]
Matchings & Set Partitions

A *perfect matching* $M$ is a graph such that every vertex is “matched” with another vertex.

For example,

```
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
```

Notation

- $M_n$ is the set of all matchings on $2n$ vertices.
- $P_n$ is the set of all set partitions on $n$ vertices.
Matchings & Set Partitions

A *perfect matching* $M$ is a graph such that every vertex is “matched” with another vertex.

For example,

A set partition may also be represented by a graph.

For example

{\{1\}, \{2, 5\}, \{3, 7\}, \{4, 12\}, \{6, 8, 11\}, \{9\}, \{10\}}

becomes

![Graph representation of set partition](image-url)
**Matchings & Set Partitions**

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For example,

$$\begin{align*}
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\end{array}
\end{align*}$$

A set partition may also be represented by a graph.

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becomes

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**Notation**

- $\mathcal{M}_n$ is set of all matchings on $2n$ vertices.
- $\mathcal{P}_n$ is set of all set partitions on $n$ vertices.
Patterns in Matchings and Set Partitions

In this context, a pattern is a certain configuration of arcs. For example consider:

- **3-crossing**
- **3-nesting**

Both contain a 3-crossing and both avoid a 3-nesting.
Patterns in Matchings and Set Partitions

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Now consider our examples:

- Both contain a 3-crossing and both avoid a 3-nesting.

Notation

If $\tau$ is a configuration let...

- $\mathcal{M}_n(\tau)$ be matchings on $2n$ vertices that avoid $\tau$.
- $\mathcal{P}_n(\tau)$ be set partitions of $[n]$ that avoid $\tau$. 
The mapping (due to C. Krattenthaler) \( \kappa : M_n \rightarrow R^\times_n \) is a bijection.

This will permit us to translate between matchings and f.r.p.
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This will permit us to translate between matchings and full rook placements.
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Matchings & Full Rook Placements

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★ This will permit us to translate between matchings and f.r.p.
Matchings & Partitions that avoid 231

In particular, \( \kappa : \mathcal{M}_n(231) \rightarrow \mathcal{M}_n(\tau) \) where \( \tau \) is the configuration:

\[
\begin{array}{c}
\text{\[231\]} \\
\end{array}
\]
Matchings & Partitions that avoid 231

In particular, $\kappa : M_n(231) \rightarrow M_n(\tau)$ where $\tau$ is the configuration:

\[
\begin{array}{c}
\text{
} \\
\end{array}
\]

In light of this the following notation makes sense...

Notation

- $M_n(231)$ is the set of all matchings that avoid $\tau$.
- $P_n(231)$ is the set of all set partitions that avoid $\tau$. 

\[\star\] The other patterns of length 3 correspond to "nice" configuration of 3 arcs as well.
For example,
- $123 \mapsto 3$-nesting.
- $321 \mapsto 3$-crossing.
Matchings & Partitions that avoid 231

In particular, \( \kappa : \mathcal{M}_n(231) \to \mathcal{M}_n(\tau) \) where \( \tau \) is the configuration:

\[
\begin{array}{c}
\text{---}
\end{array}
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Notation

- \( \mathcal{M}_n(231) \) is the set of all matchings that avoid \( \tau \).
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The other patterns of length 3 correspond to “nice” configuration of 3 arcs as well. For example,
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- \( 321 \mapsto 3\)-crossing.
Matchings & Partitions that avoid 231

Definition
In a matching $M$ a valley is the occurrence of a “closer” followed by an “opener”, i.e.,

\[ n \longrightarrow n+1 \]
Matchings & Partitions that avoid 231

Definition
In a matching $M$ a valley is the occurrence of a “closer” followed by an “opener”, i.e.,

![Diagram of a valley in a matching](image)

Lemma (J. Bloom, S. Elizalde)
Let $\tau$ be any configuration. Then, given

$$A(v, z) = \sum_{n \geq 0} \sum_{M \in \mathcal{M}_n(\tau)} v^{\text{val}(M)} z^n$$

we have

$$\sum_{n \geq 0} |\mathcal{P}_n(\tau)| z^n = \frac{1}{1 - z} A\left(\frac{1}{z}, \frac{z^2}{(1 - z)^2}\right)$$
Matchings & Partitions that avoid 231

To obtain $\sum |\mathcal{P}_n(231)|z^n$ it will suffice to have $\sum_{\mathcal{M}_n(231)} v^{\text{val}(M)} z^n$. 
Matchings & Partitions that avoid 231

To obtain $\sum |\mathcal{P}_n(231)| z^n$ it will suffice to have $\sum_\mathcal{M}_n(231) v^{\text{val}(M)} z^n$. 

\[ \begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array} \] 

\[ \begin{array}{c}
\times & \times & \times & \times \\
\times & \times & \times \\
\times & \times \\
\end{array} \] 

$\kappa$ 

\[ \begin{array}{c}
0 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 \\
2 & 1 \\
0 \\
\end{array} \] 

$\Pi$
To obtain $\sum |\mathcal{P}_n(231)| z^n$ it will suffice to have $\sum_{\mathcal{M}_n(231)} v^{\text{val}(M)} z^n$. 

Translating to generating functions:

$$\sum_{\mathcal{M}_n(231)} v^{\text{val}(M)} z^n = \sum_{\mathcal{R}_n(231)} v^{\text{val}(F)} z^n = \sum_{\mathcal{L}_n(231)} v^{\text{val}(D)} z^n.$$
Matchings & Partitions that avoid 231

To obtain $\sum |P_n(231)|z^n$ it will suffice to have $\sum_{M_n(231)} v^{\text{val}(M)} z^n$. 

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Matchings & Partitions that avoid 231

To obtain $\sum |P_n(231)| z^n$ it will suffice to have $\sum M_{n}(231)^{\text{val}(M)} z^n$. 

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Matchings & Partitions that avoid 231

To obtain \( \sum |\mathcal{P}_n(231)|z^n \) it will suffice to have \( \sum_{\mathcal{M}_n(231)} v^{\text{val}(M)}z^n \).

Translating to generating functions:

\[
\sum_{\mathcal{M}_n(231)} v^{\text{val}(M)}z^n = \sum_{\mathcal{R}_n(231)} v^{\text{val}(F)}z^n = \sum_{\mathcal{L}_n} v^{\text{val}(D)}z^n
\]
Theorem (J. Bloom, S. Elizalde)

The generating function $\sum_{n\geq 0} |P_n(231)|z^n$ is a root of the cubic polynomial

$$(z - 1)(5z^2 - 2z + 1)^2X^3$$

$$+ (-9z^5 + 54z^4 - 85z^3 + 59z^2 - 14z + 3)X^2$$

$$+ (-9z^4 + 60z^3 - 64z^2 + 13z - 3)X + (-9z^3 + 23z^2 - 4z + 1).$$

The asymptotic behavior of its coefficients is given by

$$|P_n(312)| \sim \delta n^{-5/2} \rho^n,$$

where $\delta \approx 0.061518$ and

$$\rho = \frac{3(9 + 6\sqrt{3})^{1/3}}{2 + 2(9 + 6\sqrt{3})^{1/3} - (9 + 6\sqrt{3})^{2/3}} \approx 6.97685.$$
Shape-Wilf-Equivalent Pairs

I
\{123, 213\} \sim \{132, 213\} \sim \{132, 231\} \sim \{132, 312\} \sim \{213, 231\} \sim \{213, 312\} \sim \{231, 312\} \sim \{231, 321\} \sim \{312, 321\}

II
\{123, 231\}

III
\{123, 312\}

IV
\{123, 321\}

V
\{213, 321\}

VI
\{123, 132\}

VII
\{132, 321\}

Class

Matchings

IV
\frac{1}{1 - 5z + 2z^2 - 10z^3 + 32z^4 - 37z^5 + 12z^6} \left(1 - z \left(1 - 10z - 31z^2 + 30z^3 + z^4\right)\right)

V
\text{Unknown}

VI & VII
\text{Unknown}
### Shape-Wilf-Equivalent Pairs

<table>
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<th>Set partitions</th>
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</table>
| I     | \[
\frac{4}{3 + \sqrt{1 - 8z}}
\]
|       | \[
\frac{2 - 3z + z^2 - z\sqrt{1 - 6z + z^2}}{2(1 - 3z + 3z^2)}
\] | |
| II & III | Solution of a cubic | Solution of a cubic |
| IV    | \[
\frac{1 - 5z + 2z^2}{1 - 6z + 5z^2}
\]
|       | \[
\frac{1 - 10z + 32z^2 - 37z^3 + 12z^4}{(1 - z)(1 - 10z + 31z^2 - 30z^3 + z^4)}
\] | |
| V     | Solution of a functional equation | Unknown |
| VI & VII | Unknown | Unknown |