From pattern avoidance to rectangular Young tableaux: two new results

Jonathan S. Bloom

Dartmouth College

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2. He stated a conjecture of Roby-Propp about homomesy, rectangular Young tableaux, and promotion
3. Then we prove it!

Part 1: Pattern Avoidance

Classical Pattern Avoidance

Definition

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Notation

- *Denote by $S_n(\tau)$ the set of all $\pi \in S_n$ that avoids τ.*

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- No general method for determining Wilf-equivalence
  - Finding one is the Holy Grail!

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- What is that?

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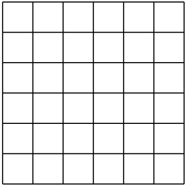

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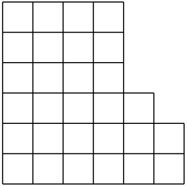

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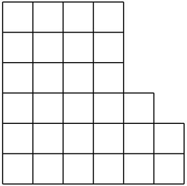

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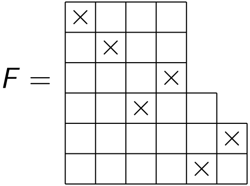

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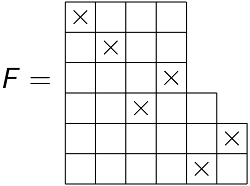

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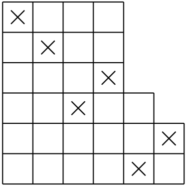

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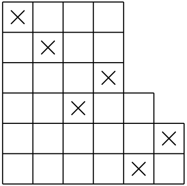

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Notation

1. $\mathcal{R}_F =$ set of all f.r.p. on fixed board F
2. $\mathcal{R}_n =$ set of all f.r.p. with n rooks (different boards)
    - Analogous to S_n

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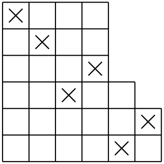

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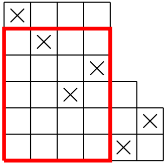

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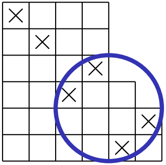

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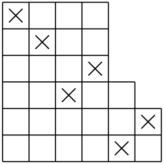

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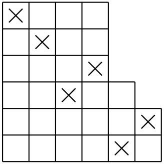

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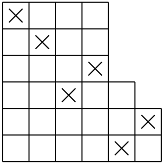

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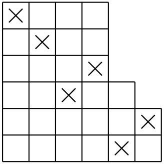

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- $\mathcal{R}_n(\tau)$ = *subset of* $\mathcal{R}_n$ *that avoid* τ
  - *Analogous to* $S_n(\tau)$

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Observe:

- shape-Wilf-equivalence → classical Wilf-equivalence.
  - If F is $n \times n$ square board, then $\mathcal{R}_F(\tau) = S_n(\tau)$.

Patterns of Length 3

There are 3 shape-Wilf-equivalence classes:

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  - Yields many enumerative results

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- First, we define a bijection

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- Then, we define another bijection
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- Finally, we show that $\mathcal{L}_n(231) \leftrightarrow \mathcal{L}_n(312)$

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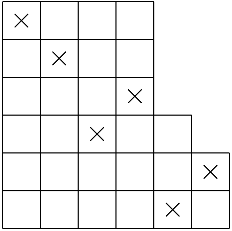
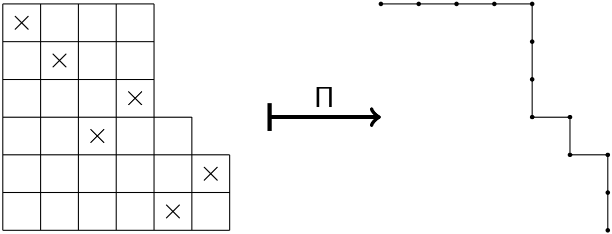


$\mathcal{R}_F(231)$

$$\xmapsto{\Pi}$$

The Bijection Π



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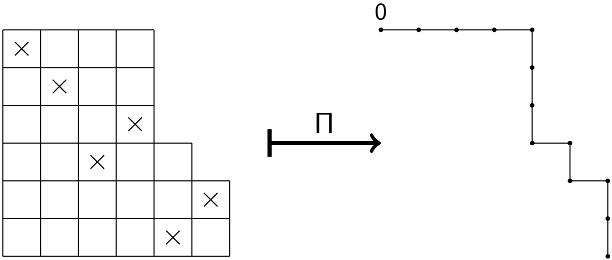

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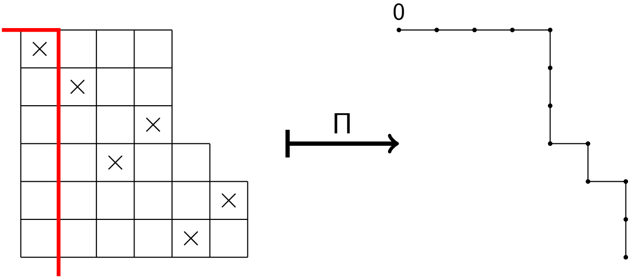

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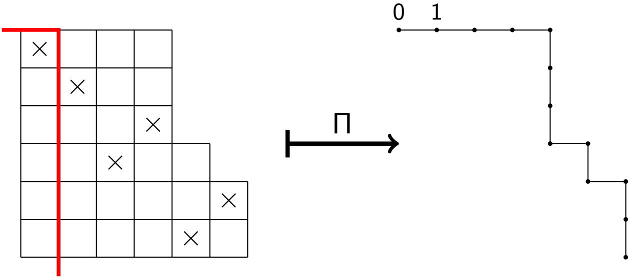

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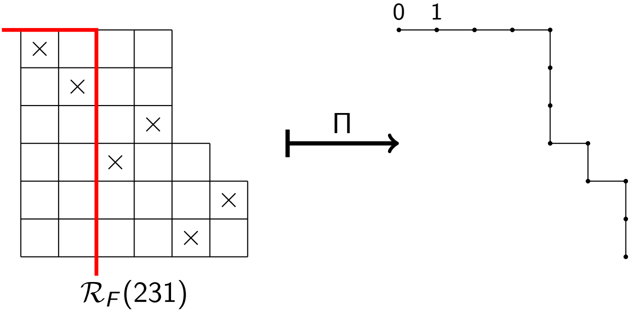

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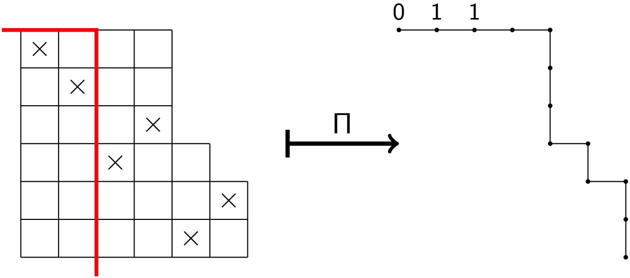

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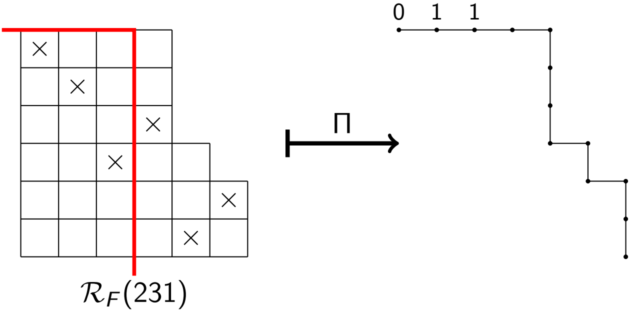

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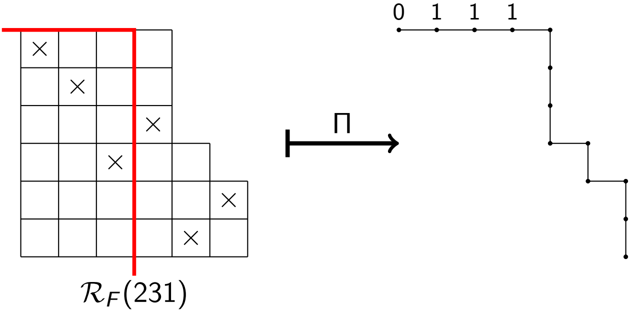

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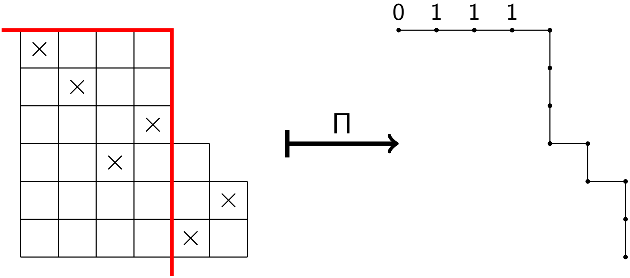

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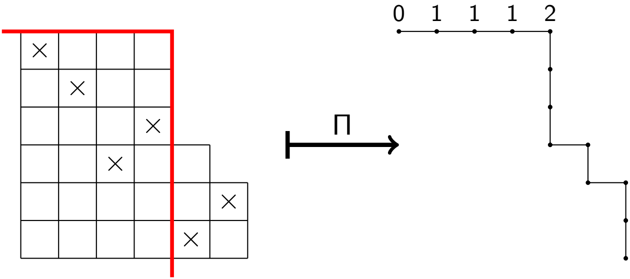

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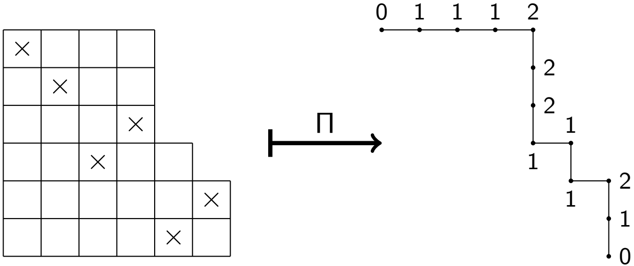

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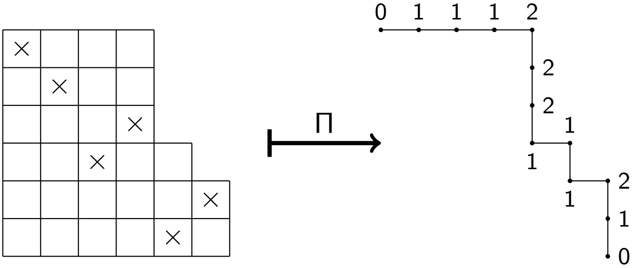
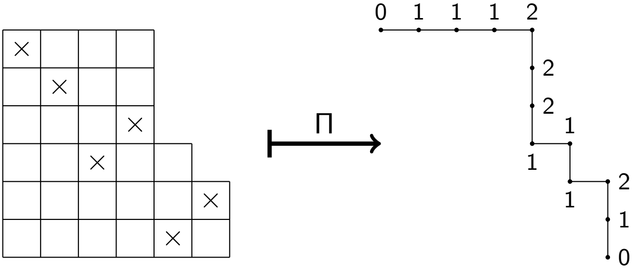


$\mathcal{R}_F(231)$

$\mathcal{L}_F(231)$

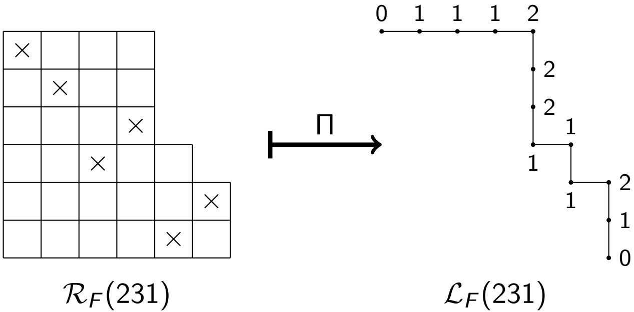
The Bijection Π



$\mathcal{R}_F(231)$ $\mathcal{L}_F(231)$

Defining Properties of $\mathcal{L}_F(231)$

The Bijection Π



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- Monotonicity:
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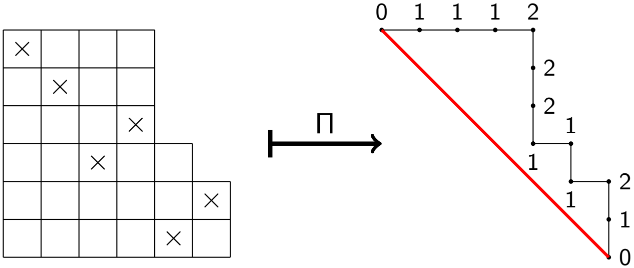
The Bijection Π



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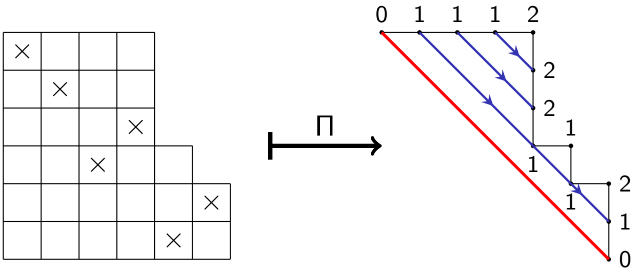
The Bijection Π



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Defining Properties of $\mathcal{L}_F(231)$

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- Zero Condition:
  - All zeros are along the main diagonal (red line)
- Diagonal Property:
  - Upper $\leq$ Lower

The Bijection Π

Theorem (Bloom-Saracino '11)

The mapping

$$\Pi : \mathcal{R}_F(231) \to \mathcal{L}_F(231)$$

is a bijection.

The Bijection Π

Theorem (Bloom-Saracino '11)

The mapping

$$\Pi : \mathcal{R}_F(231) \to \mathcal{L}_F(231)$$

is a bijection. Further, an analogous mapping

$$\Theta : \mathcal{R}_F(312) \to \mathcal{L}_F(312),$$

is bijective. Here $\mathcal{L}_F(312) =$ *the set of labelings with the* **reverse** *diagonal property:*

$$\textit{Upper} \geq \textit{Lower}$$

Proof of $231 \sim_s 312$

Corollary (Bloom-Saracino '11)

There exists a (simple) bijection $\mathcal{R}_F(231) \longleftrightarrow \mathcal{R}_F(312)$.

Proof of 231 $\sim_s$ 312

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Proof by Example:

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Proof by Example:

$$\begin{tikzcd}
\mathcal{R}_F(231) \arrow[r, leftrightarrow, "\Pi"] & \mathcal{L}_F(231) \arrow[r, leftrightarrow, "??"] & \mathcal{L}_F(312) \arrow[r, leftrightarrow, "\Theta"] & \mathcal{R}_F(312)
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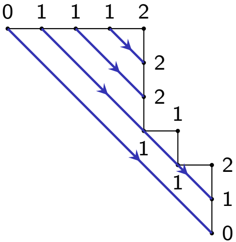

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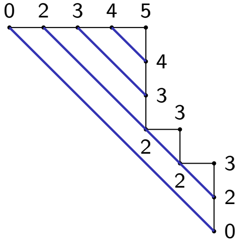
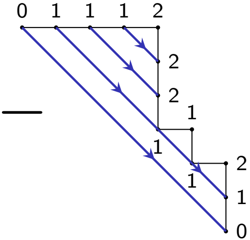


(Canonical Labeling)

—



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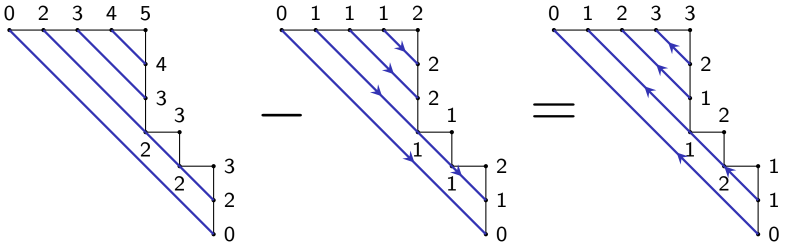

(Canonical Labeling) $\mathcal{L}_F(231)$ $\mathcal{L}_F(312)$

Enumerative Results

Enumerative Results

Theorem (Bloom-Elizalde '13)

$$\sum_{n\geq 0} |\mathcal{R}_n(231)| z^n = \sum_{n\geq 0} |\mathcal{L}_n(231)| z^n = \frac{54z}{1+36z-(1-12z)^{3/2}}.$$

Further, we obtain

$$|\mathcal{R}_n(231)| \sim \frac{3^3}{2^5\sqrt{\pi n^5}} 12^n.$$

Enumerative Results: 2314−Avoiding Permutations

In 1997 M. Bóna proved the following celebrated result:

$$\sum_{n\geq 0} |S_n(2314)| z^n = \frac{32z}{1+20z-8z^2-(1-8z)^{3/2}}.$$

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- We provide one using Π

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First, we view any $\pi \in S_n(2314)$ as a f.r.p. on a **minimal** Ferrers board.

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×						
		×				
			×			
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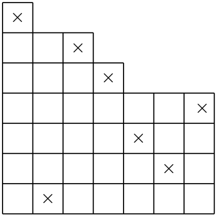

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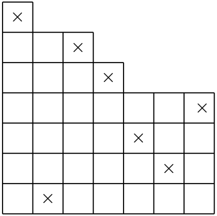

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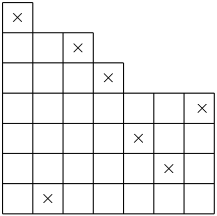

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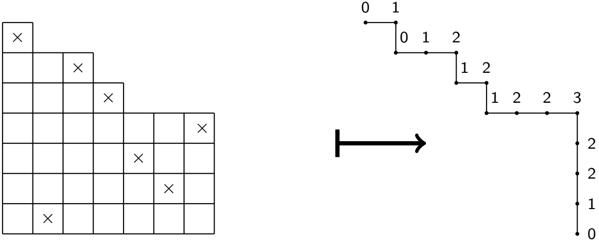

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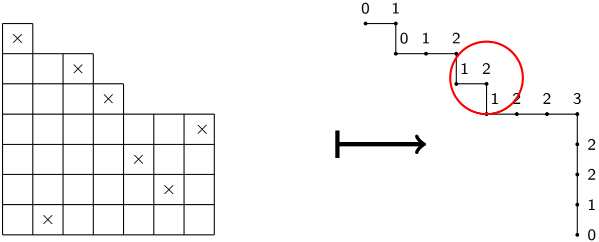

Observe

- This f.r.p. is in $\mathcal{R}_n(231)$
    - Apply Π!
- The resulting labels are characterized by the **peak property**
    - Around a peak we have: $a, a+1, a$.

Enumerative Results: 2314−Avoiding Permutations

Lemma (Bloom-Elizalde '13)

Our bijection $\Pi : \mathcal{R}_n(231) \to \mathcal{L}_n(231)$ *induces a bijection*

$$\Pi^{\times} : S_n(2314) \to \mathcal{L}_n^{\times}(231),$$

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Doing so we obtain Bóna's result:

$$\sum_{n\geq 0} |S_n(2314)| z^n = \sum_{n\geq 0} |\mathcal{L}_n^{\times}(312)| z^n = \frac{32z}{1+20z-8z^2-(1-8z)^{3/2}}.$$

Enumerative Results

In June 2013, D. Callan proved that

$$\sum_{n\geq 0} |S_n(2314, 1234)| z^n = \frac{1}{1 - zC(zC(z))},$$

where $C(z)$ is the generating function for the Catalan numbers.

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The answer is **YES!**

- Using Π the proof is < 1 page.

Part 2: Homomesy

What is Homomesy?

Definition

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If we have

- X - a set of combinatorial objects

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If we have

- X - a set of combinatorial objects
- G - a group acting on X
- $f : X \to \mathbb{R}$ (a "statistic"),

then we say the triple (X, G, f) is *homomesic* if there is some constant C such that

$$\frac{1}{|\mathcal{O}|} \sum_{x \in \mathcal{O}} f(x) = C$$

where $\mathcal{O}$ is any orbit.

Our set X: Rectangular Young Tableaux

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Definition (By Example)

A **rectangular Young tableau** is a rectangular array of N boxes

1	2	3	7
4	5	6	8

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Our set X: Rectangular Young Tableaux

Definition (By Example)

A **rectangular Young tableau** is a rectangular array of N boxes

1	2	3	7
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- that contains the numbers $1 \dots N$
- with rows/columns strictly increasing

Our Group Action: Promotion

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Consider the mapping

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 & 8 \\ \hline \end{array}$$

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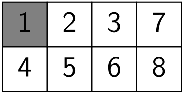

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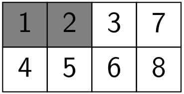

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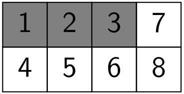

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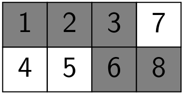

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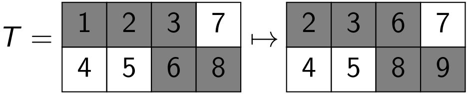

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 & 8 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 2 & 3 & 6 & 7 \\ \hline 4 & 5 & 8 & 9 \\ \hline \end{array}$$

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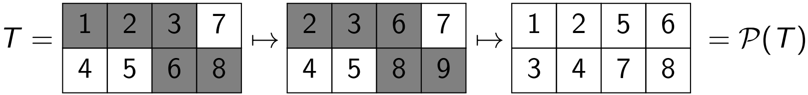

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Our Group Action: Promotion

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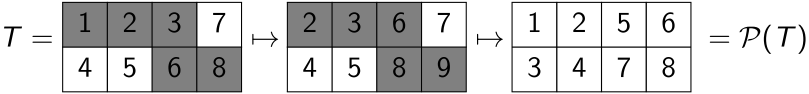

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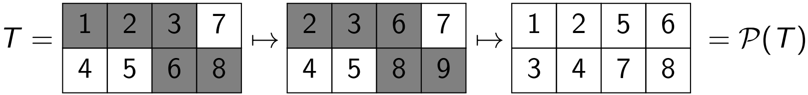

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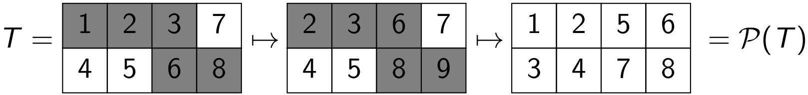

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Propp-Roby Conjecture

Consider the orbit of T under promotion:

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&7\\\hline 4&5&6&8\\\hline\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&7&8\\\hline\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2&6&7&8\\\hline\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6&7&8\\\hline\end{array}$$

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Pick a box B

- Let B^* be the corresponding box (under 180°-rotation)

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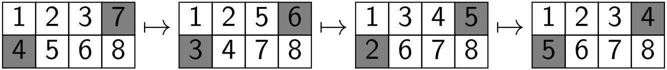

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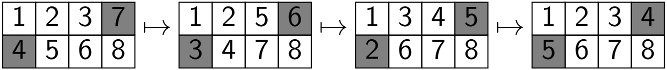

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Pick a box B

- Let B^* be the corresponding box (under 180°-rotation)
- The average value in these two boxes is:

$$\frac{(4+7)+(3+6)+(2+5)+(5+4)}{4} = \frac{36}{4} = 8+1$$

Propp-Roby Conjecture

Conjecture (Propp-Roby)

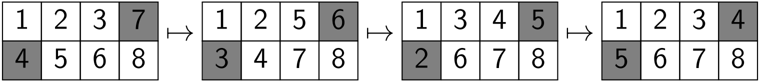
Let T be a rectangular Young Tableau with N boxes. If B is any box and B^* is its corresponding box (under 180°-rotation) then their average value over the orbit

$$T \mapsto \mathcal{P}(T) \mapsto \mathcal{P}^2(T) \mapsto \cdots$$

is always $N + 1$.

Propp-Roby Conjecture: A Closer Look

Again consider the orbit of T:



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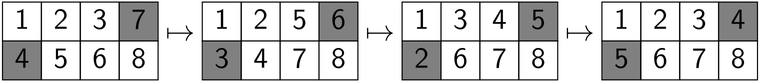

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Observe the distributions in B and B^*:

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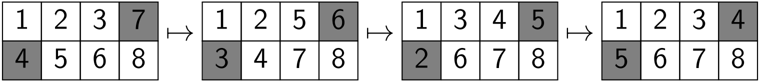

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$$\text{Dist}(B) = \{7,\ 6,\ 5,\ 4\}$$

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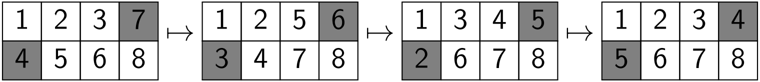

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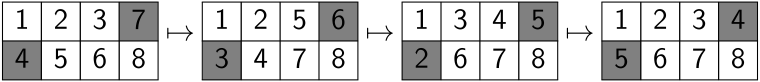

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Theorem (Bloom-Pechenik-Saracino)

Let T be a rectangular Young Tableau with N boxes. If B is any box and B^* is the corresponding box then

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Growth Diagrams

We encode our tableau as a sequence of partitions:

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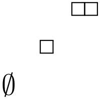

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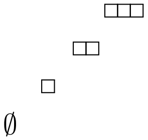

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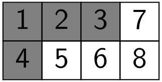

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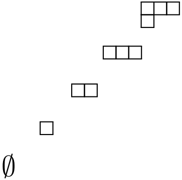

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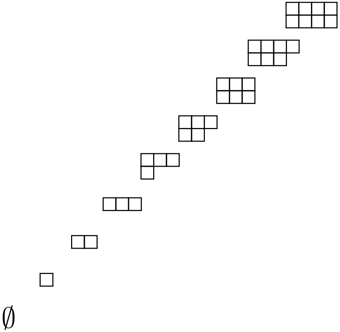

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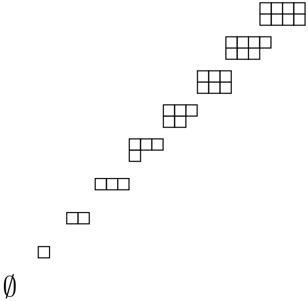

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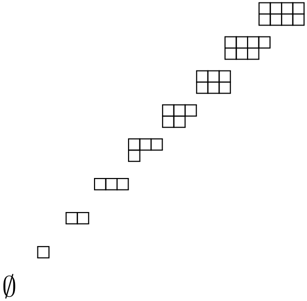

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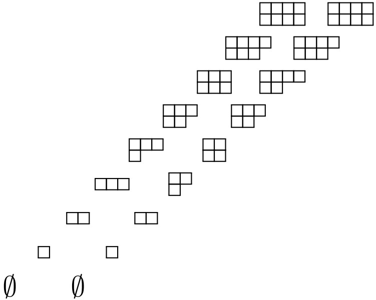

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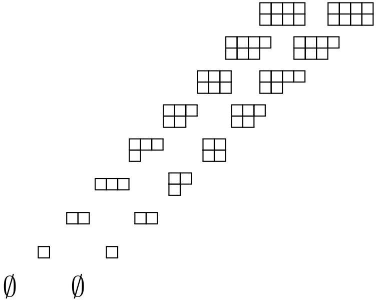

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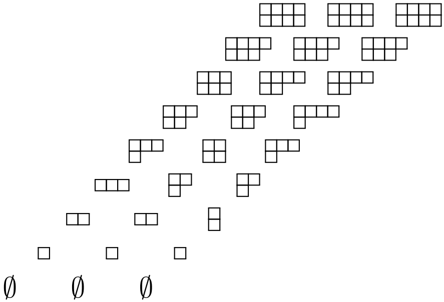

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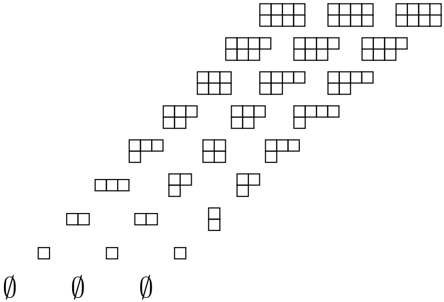

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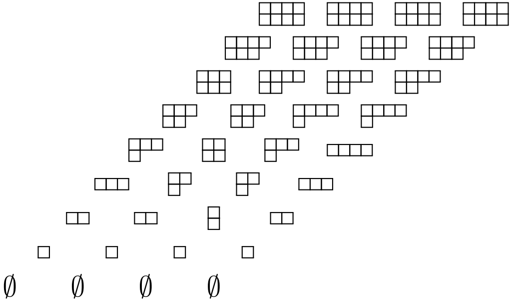

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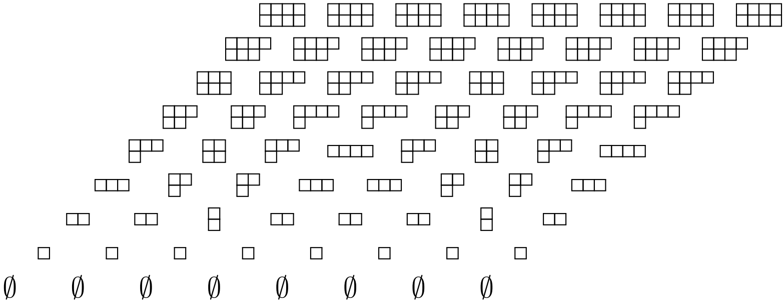

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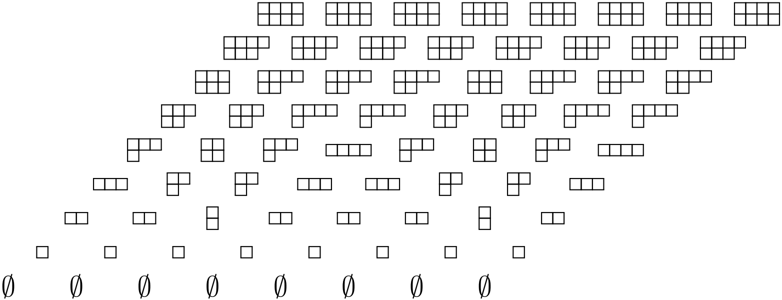
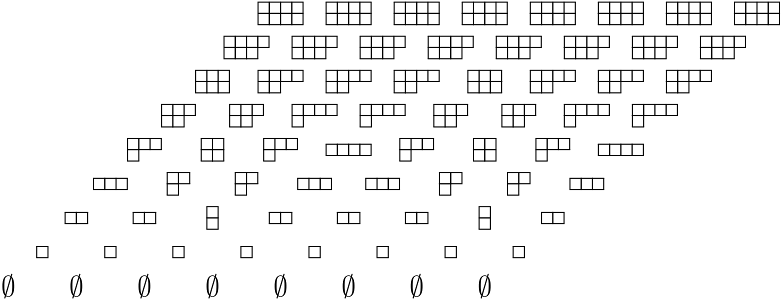
becomes



- This is Fomin's growth diagram.

Growth Diagrams

Some KEY fact about growth diagram:



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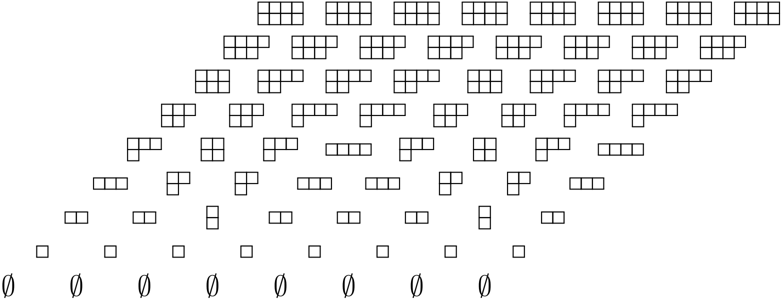

- The diagonals encode the orbit of T

Growth Diagrams

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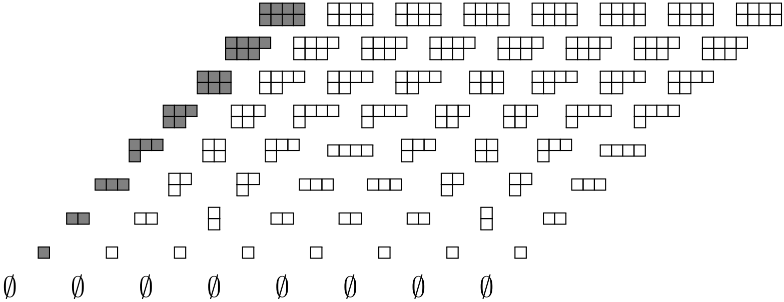

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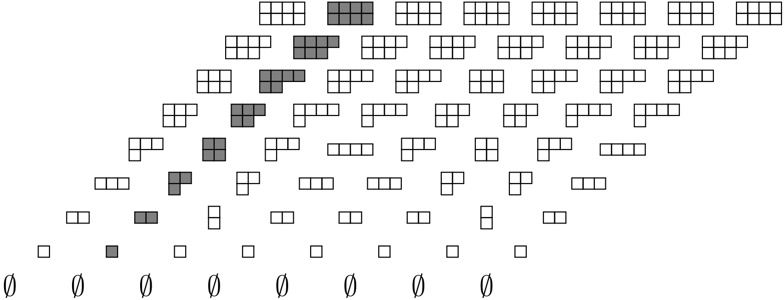

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Growth Diagrams

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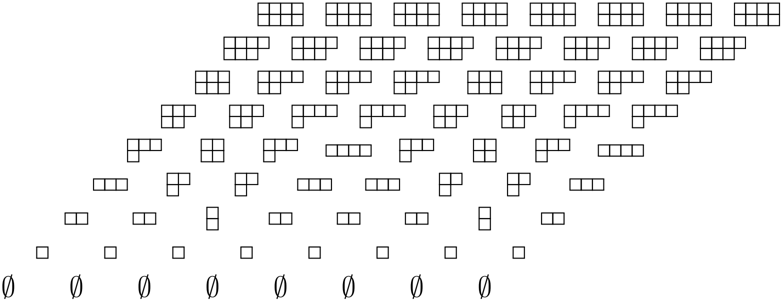

- The diagonals encode the orbit of T
- The anti-diagonals encode the orbit of T^*

Growth Diagrams

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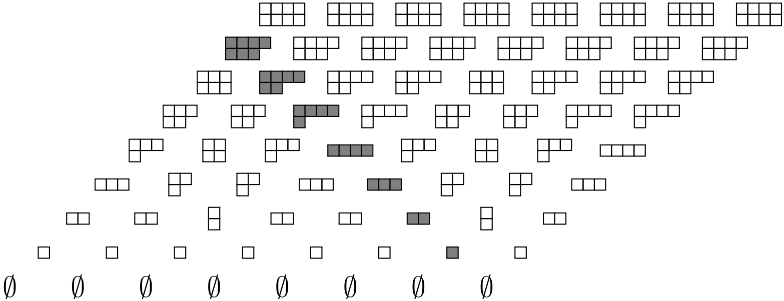

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Growth Diagrams

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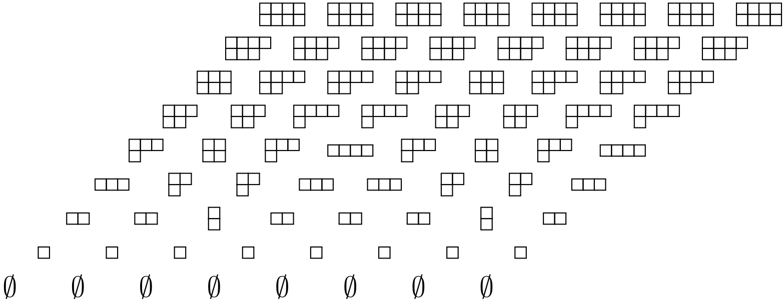

- The diagonals encode the orbit of T
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- The addition of a box B on level k means:

Growth Diagrams

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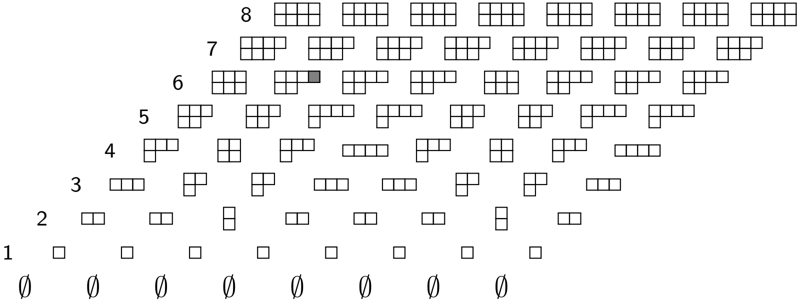

- The diagonals encode the orbit of T
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Growth Diagrams

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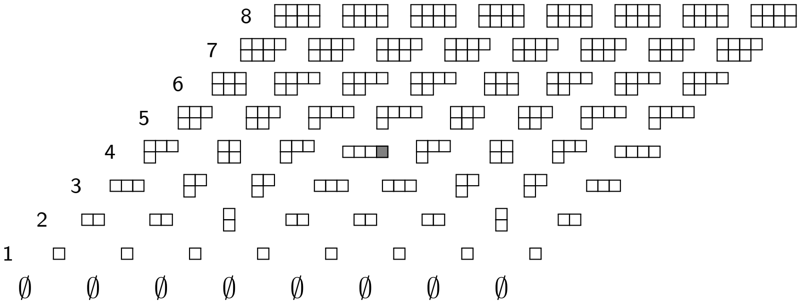

- The diagonals encode the orbit of T
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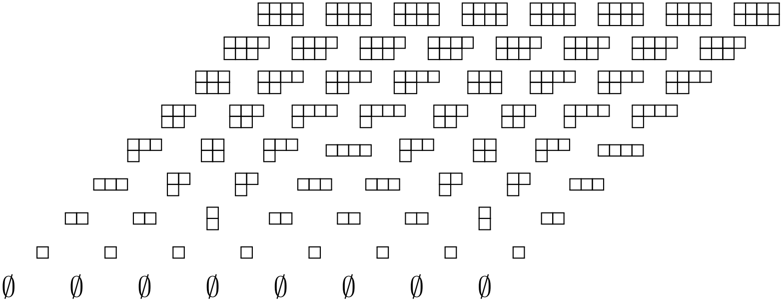
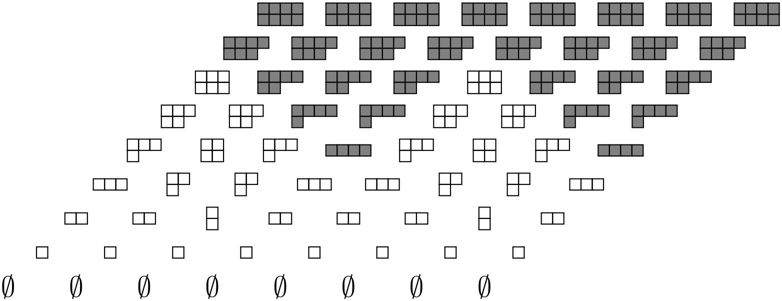
- The addition of a box B on level k means:
    - Along a diagonal: $k \in \mathrm{Dist}_T(B)$
    - Along an anti-diagonal: $k \in \mathrm{Dist}_{T^*}(B)$

Our proof

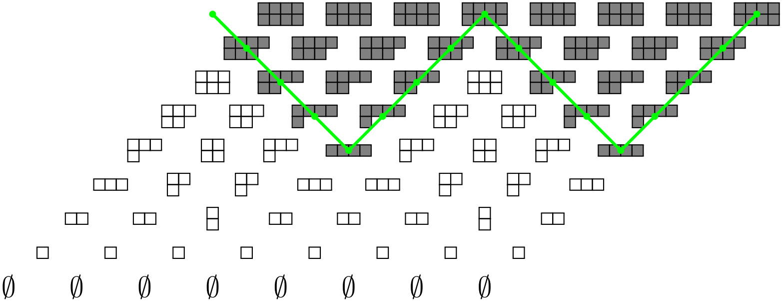


Our proof



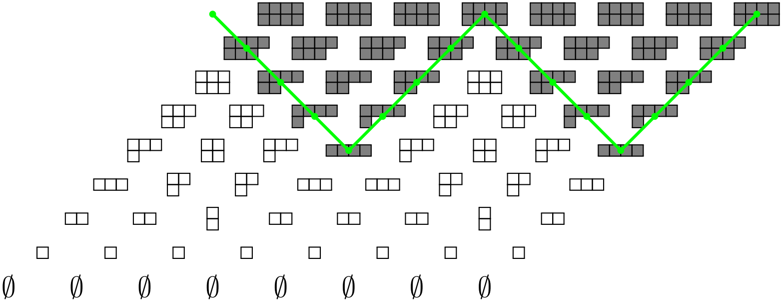
- Shade all partitions containing B

Our proof



- Shade all partitions containing B
  - This carves out a Dyck path
  - Up-steps $\longleftrightarrow$ down-steps

Our proof



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- Down-step on level $k \longleftrightarrow k \in \mathrm{Dist}_T(B)$

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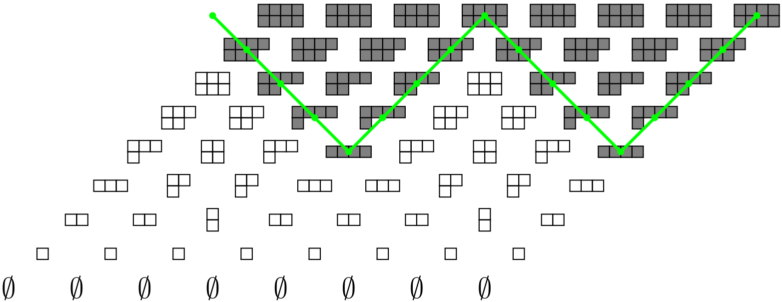

- Shade all partitions containing B
  - This carves out a Dyck path
  - Up-steps $\longleftrightarrow$ down-steps
- Down-step on level $k \longleftrightarrow k \in \mathrm{Dist}_T(B)$
- Every up-step on level $k \longleftrightarrow k \in \mathrm{Dist}_{T^*}(B)$

Thank You!