Modified Growth Diagrams and the BWX Map $\phi^*$

Jonathan Bloom

Dartmouth College

Saint Michael’s College July, 2011
Permutations and Pattern Avoidance

Consider a permutation $\sigma \in S_n$ as a word in the alphabet $\{1, 2, 3, \ldots, n\}$.

For example we write $\sigma = 4 \ 5 \ 3 \ 1 \ 2$ for the permutation (in 2-line notation)

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
4 & 5 & 3 & 1 & 2
\end{pmatrix}
\]
Consider a permutation $\sigma \in S_n$ as a word in the alphabet $\{1, 2, 3, \ldots, n\}$. 
Consider a permutation $\sigma \in S_n$ as a word in the alphabet \{1, 2, 3, \ldots, n\}.

For example we write

$$\sigma = 4\ 5\ 3\ 1\ 2$$

for the permutation (in 2-line notation)

$$\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
4 & 5 & 3 & 1 & 2
\end{pmatrix}$$
We say a permutation $\sigma \in S_n$ contains a pattern $\tau \in S_k$ if $\sigma$ contains a subsequence which is order-isomorphic to $\tau$.

For example $\sigma = 4 \ 5 \ 3 \ 1 \ 2$ contains $\tau = 3 \ 2 \ 1$ but does not contain $\tau = 1 \ 2 \ 3$.

We write $S_n(\tau)$ for all permutations of length $n$ which avoid $\tau$. 
Permutations and Pattern Avoidance

We say a permutation $\sigma \in S_n$ contains a pattern $\tau \in S_k$ if $\sigma$ contains a subsequence which is order-isomorphic to $\tau$. For example $\sigma = 4 \ 5 \ 3 \ 1 \ 2$ contains $\tau = 3 \ 2 \ 1$ but does not contain $\tau = 1 \ 2 \ 3$. We write $S_n(\tau)$ for all permutations of length $n$ which avoid $\tau$. 
Permutations and Pattern Avoidance

We say a permutation \( \sigma \in S_n \) contains a pattern \( \tau \in S_k \) if \( \sigma \) contains a subsequence which is order-isomorphic to \( \tau \).

For example

\[
\sigma = 4 \ 5 \ 3 \ 1 \ 2 \ \text{contains} \ \tau = 3 \ 2 \ 1
\]

but does not contain \( \tau = 1 \ 2 \ 3 \).
Permutations and Pattern Avoidance

We say a permutation $\sigma \in S_n$ contains a pattern $\tau \in S_k$ if $\sigma$ contains a subsequence which is order-isomorphic to $\tau$.

For example

$$\sigma = 4 \ 5 \ 3 \ 1 \ 2 \text{ contains } \tau = 3 \ 2 \ 1$$

but does not contain $\tau = 1 \ 2 \ 3$.

We write $S_n(\tau)$ for all permutations of length $n$ which avoid $\tau$. 
An important problem in pattern avoidance is to determine when $|S_n(τ)| = |S_n(ρ)|$ for all $n$. For two distinct patterns $τ$ and $ρ$, ▶ in this case we say $τ$ is Wilf-Equivalent to $ρ$.

A classic result in the field is that $|S_n(τ)| = C_n = \frac{1}{n+1} \binom{2n}{n}$ for all $τ \in S_3$. 
An important problem in pattern avoidance is to determine when
\[ |S_n(\tau)| = |S_n(\rho)| \]
for all \( n \). for two distinct patterns \( \tau \) and \( \rho \).
An important problem in pattern avoidance is to determine when

$$|S_n(\tau)| = |S_n(\rho)|$$

for all $n$. for two distinct patterns $\tau$ and $\rho$.

$\Rightarrow$ In this case we say $\tau$ is Wilf-Equivalent to $\rho$. 

A classic result in the field is that

$$|S_n(\tau)| = C_n = \frac{1}{n+1}(2^n)$$

for all $\tau \in S_3$. 

An important problem in pattern avoidance is to determine when

$$|S_n(\tau)| = |S_n(\rho)|$$

for all $n$. for two distinct patterns $\tau$ and $\rho$.

- In this case we say $\tau$ is Wilf-Equivalent to $\rho$.

A classic result in the field is that

$$|S_n(\tau)| = C_n = \frac{1}{n+1} \binom{2n}{n}$$

for all $\tau \in S_3$. 

Permutations and Pattern Avoidance
Another classic result due to Backelin, West, Xin (BWX) is:

$$S_n(12...k\rho) = S_n(k...1\rho)$$

for all $n$ where $\rho$ is a permutation of $\{k+1,...,k+l\}$.

An important tool in their proof is the map $\phi^*$:

$$S_n \rightarrow S_n(k...1)$$

which is the focus of this talk.
Another classic result due to Backelin, West, Xin (BWX) is:

$$|S_n(12\ldots k\rho)| = |S_n(k\ldots 1\rho)| \text{ for all } n$$

where $\rho$ is an permutation of $\{k + 1, \ldots, k + l\}$. 
Another classic result due to Backelin, West, Xin (BWX) is:

$$|S_n(12\ldots k\rho)| = |S_n(k\ldots 1\rho)|$$ for all $n$

where $\rho$ is an permutation of $\{k + 1, \ldots, k + l\}$.

An important tool in their proof is the map

$$\phi^* : S_n \to S_n(k\ldots 1)$$

which is the focus of this talk.
Definition of the BWX map $\phi^*$

First we define the (intermediate) map

$$\phi : S_n \rightarrow S_n$$

which is implicitly dependent on some fixed $k > 2$. 

Key Idea: $\phi$ removes the smallest $k \ldots 1$ pattern.
Definition of the BWX map $\phi^*$

First we define the (intermediate) map

$$\phi : S_n \to S_n$$

which is implicitly dependent on some fixed $k > 2$.

For any $\sigma \in S_n$
Definition of the BWX map $\phi^*$

First we define the (intermediate) map

$$\phi : S_n \rightarrow S_n$$

which is implicitly dependent on some fixed $k > 2$.

For any $\sigma \in S_n$

- Take the smallest $k \ldots 1$-pattern in $\sigma$ and cycle these entries forward leaving all other fixed.
Definition of the BWX map $\phi^*$

First we define the (intermediate) map

$$\phi : S_n \to S_n$$

which is implicitly dependent on some fixed $k > 2$.

For any $\sigma \in S_n$

- Take the smallest $k \ldots 1$-pattern in $\sigma$ and cycle these entries forward leaving all other fixed.

For example if $k = 3$ and $\sigma = 4 \ 5 \ 3 \ 1 \ 2$ then

$$\phi(\sigma) = 3 \ 5 \ 1 \ 4 \ 2$$
Definition of the BWX map $\phi^*$

First we define the (intermediate) map

$$\phi : S_n \rightarrow S_n$$

which is implicitly dependent on some fixed $k > 2$.

For any $\sigma \in S_n$

- Take the smallest $k \ldots 1$-pattern in $\sigma$ and cycle these entries forward leaving all other fixed.

For example if $k = 3$ and $\sigma = 4 \ 5 \ 3 \ 1 \ 2$ then

$$\phi(\sigma) = 3 \ 5 \ 1 \ 4 \ 2$$

**Key Idea:** $\phi$ removes the smallest $k \ldots 1$ pattern.
Definition of the BWX map $\phi^*$

Now the map of interest

$$\phi^* : S_n \rightarrow S_n(k \ldots 1)$$
Definition of the BWX map $\phi^*$

Now the map of interest

$$\phi^* : S_n \rightarrow S_n(k \ldots 1)$$

is obtained by repeatedly applying the map $\phi$ until no $(k \ldots 1)$-pattern remains.
The commutativity of $\phi^*$

It was first observed by Bousquet-Mélou and Steingrímsson that

$$\phi^*(\sigma^{-1}) = \phi^*(\sigma)^{-1}$$
The commutativity of $\phi^*$

It was first observed by Bousquet-Méléou and Steingrímsson that

$$\phi^*(\sigma^{-1}) = \phi^*(\sigma)^{-1}$$

- Their proof is long and difficult.
The commutativity of $\phi^*$

It was first observed by Bousquet-Mélou and Steingrímsson that

$$\phi^*(\sigma^{-1}) = \phi^*(\sigma)^{-1}$$

- Their proof is long and difficult.
- They ask for an alternative description of the map $\phi^*$ “on which the commutation theorem would become obvious.”
The commutativity of $\phi^*$

It was first observed by Bousquet-Mélou and Steingrímsson that

$$\phi^*(\sigma^{-1}) = \phi^*(\sigma)^{-1}$$

- Their proof is long and difficult.
- They ask for an alternative description of the map $\phi^*$ “on which the commutation theorem would become obvious.”

Later, Krattenthaler published a bijection based on the standard Growth Diagram Algorithm (GDA) which is similar in functionality to $\phi^*$ and trivially commutes with inverses.
The commutativity of $\phi^*$

It was first observed by Bousquet-Mélou and Steingrímsson that

$$\phi^*(\sigma^{-1}) = \phi^*(\sigma)^{-1}$$

- Their proof is long and difficult.
- They ask for an alternative description of the map $\phi^*$ “on which the commutation theorem would become obvious.”

Later, Krattenthaler published a bijection based on the standard Growth Diagram Algorithm (GDA) which is similar in functionality to $\phi^*$ and trivially commutes with inverses.

- He explicitly ask for a connection between $\phi^*$ and the GDA.
Ferrers Boards & Placements

Definition (Informal): A Ferrers Board $F$ is an array of squares obtained by removing some "northeast chunk" from the $n \times n$ array leaving a staircase shape.

Definition: A rook placement $P$ on a Ferrers Board $F$ is an arrangement of dots with no two in the same row or column.
Definition (Informal): A Ferrers Board $F$ is an array of squares obtained by removing some “northeast chunk” from the $n \times n$ array of squares leaving a staircase shape.
**Definition (Informal):** A Ferrers Board $F$ is an array of squares obtained by removing some “northeast chunk” from the $n \times n$ array of squares leaving a staircase shape.

**Definition:** A rook placement $P$ on a Ferrers Board $F$ is an arrangement of dots with no two in the same row or column.
Ferrers Boards & Placements

**Definition (Informal):** A Ferrers Board $F$ is an array of squares obtained by removing some “northeast chunk” from the $n \times n$ array of squares leaving a staircase shape.

![Diagram of a Ferrers Board]

**Definition:** A rook placement $P$ on a Ferrers Board $F$ is an arrangement of dots with no two in the same row or column.
Motivation for the Reformulation of $\phi^*$

Recall the Schensted correspondence $S_n \leftrightarrow (P, Q)$ where $P$ and $Q$ are tableaux of the same shape.

For example, $4, 5, 3, 1, 2 \leftrightarrow (1, 2, 3, 5, 4, 1, 2, 6, 8, 7)$ where the tableaux have common shape 221.

**Theorem**: The length of the longest decreasing subsequence in a permutation is the number of parts in its corresponding shape.

$\triangleright$ 4, 5, 3, 1, 2 is longest and likewise 221 has 3 parts.

**Key Idea**: $\phi^*$ removes $k$ patterns $\leftrightarrow$ force shape to have $< k$ parts.
Motivation for the Reformulation of $\phi^*$

Recall the Schensted correspondence $S_n \leftrightarrow (P, Q)$ where $P$ and $Q$ are tableaux of the same shape.

For example

\[
4 \ 5 \ 3 \ 1 \ 2 \ \leftrightarrow \ (1 \ 2 \ 3, 1 \ 2 \ 6 \ 8 \ 7)
\]

where the tableaux have common shape 221.

Theorem:
The length of the longest decreasing subsequence in a permutation is the number of parts in its corresponding shape.

$\Rightarrow 4 \ 5 \ 3 \ 1 \ 2$ is longest and likewise 221 has 3 parts.

Key Idea

$\phi^*$ removes $k$ ..., 1 patterns $\leftrightarrow$ force shape to have < $k$ parts.
Motivation for the Reformulation of $\phi^*$

Recall the Schensted correspondence $S_n \longleftrightarrow (P, Q)$ where $P$ and $Q$ are tableaux of the same shape.

For example

$$4 \ 5 \ 3 \ 1 \ 2 \quad \longleftrightarrow \quad \begin{pmatrix} 1 & 2 \\ 3 & 5 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 6 & 8 \\ 7 \end{pmatrix}$$

where the tableaux have common shape 221.
Motivation for the Reformulation of $\phi^*$

Recall the Schensted correspondence $S_n \leftrightarrow (P, Q)$ where $P$ and $Q$ are tableaux of the same shape.

For example

$$4 \ 5 \ 3 \ 1 \ 2 \leftrightarrow \left( \begin{array}{cc}
1 & 2 \\
3 & 5 \\
4 & \end{array}, \begin{array}{cc}
1 & 2 \\
6 & 8 \\
7 & \end{array} \right)$$

where the tableaux have common shape 221.

**Theorem:** The length of the longest decreasing subsequence in a permutation is the number of parts in its corresponding shape.

- $4 \ 5 \ 3 \ 1 \ 2$ is longest and likewise 221 has 3 parts.
Motivation for the Reformulation of $\phi^*$

Recall the Schensted correspondence $S_n \leftrightarrow (P, Q)$ where $P$ and $Q$ are tableaux of the same shape.

For example

$4 \ 5 \ 3 \ 1 \ 2 \ \leftrightarrow \ \left( \begin{array}{c} 1 \ 2 \\ 3 \ 5 \\ 4 \end{array} , \ \begin{array}{c} 1 \ 2 \\ 6 \ 8 \\ 7 \end{array} \right)$

where the tableaux have common shape 221.

**Theorem:** The length of the longest decreasing subsequence in a permutation is the number of parts in its corresponding shape.

- $4 \ 5 \ 3 \ 1 \ 2$ is longest and likewise 221 has 3 parts.

**Key Idea**

$\phi^*$ removes $k \ldots 1$ patterns $\leftrightarrow$ force shape to have $< k$ parts.
Fomin’s Growth Diagram Construction
Fomin’s Growth Diagram Construction

Each corner of the Ferrers board is labeled a partition which is the shape of the permutation southwest of that corner.
Local Rules for Growth Diagrams

Start by assigning the empty partition $\emptyset$ on the left and bottom edges of $F$.
Local Rules for Growth Diagrams

- Start by assigning the empty partition $\emptyset$ on the left and bottom edges of $F$. 
Local Rules for Growth Diagrams

- Start by assigning the empty partition $\emptyset$ on the left and bottom edges of $F$. 
Local Rules for Growth Diagrams

Given partitions
Local Rules for Growth Diagrams

Given partitions

if $SE \neq NW$

$21 = SE \cup NW$

Key Idea: Only the last rule can increase the number of parts of a partition.
Local Rules for Growth Diagrams

Given partitions

if $SE \neq NW$

if $SW = NW = SE$

Key Idea: Only the last rule can increase the number of parts of a partition.
Local Rules for Growth Diagrams

Given partitions

- If $SE \neq NW$
  
  \[
  \begin{array}{c}
  1 \\
  \end{array} \quad \begin{array}{c}
  11 \\
  \end{array} = \begin{array}{c}
  \text{SE} \\
  \end{array} \cup \begin{array}{c}
  \text{NW} \\
  \end{array}
  \]

- If $SW = NW = SE$

  \[
  \begin{array}{c}
  11 \\
  \end{array} \quad \begin{array}{c}
  11 \\
  \end{array} = \begin{array}{c}
  \text{NW} \\
  \end{array}
  \]

  \[
  \begin{array}{c}
  11 \\
  \end{array} \quad \begin{array}{c}
  11 \\
  \end{array} = \begin{array}{c}
  \text{NW} + 1 \text{ top row} \\
  \end{array}
  \]

- If $SW \neq NW = SE$

  \[
  \begin{array}{c}
  1 \\
  \end{array} \quad \begin{array}{c}
  11 \\
  \end{array} = \begin{array}{c}
  \text{NW} + 001 \\
  \end{array}
  \]

\textbf{Key Idea:} Only the last rule can increase the number of parts of a partition.
Local Rules for Growth Diagrams

\[ \text{Def:} \quad \text{seq}(P, F) \text{ denote the sequence of partitions along the } \text{"staircase".} \]

\[ \text{seq}(P, F) := (\emptyset, 1, 2, 3, 2, 3, 4, 3, 31, 21, 211, 221, \ldots, 1, \emptyset) \]

\[ \text{Theorem:} \quad \text{seq}(P, F) \text{ uniquely determines } P. \]
Local Rules for Growth Diagrams

Def: Let \( \text{seq}(P,F) \) denote the sequence of partitions along the "staircase".

\[
\text{seq}(P,F) := (\emptyset, 1, 2, 3, 3, 3, 4, 3, 3, 31, 21, 211, 221, \ldots, 1, \emptyset)
\]

Theorem: \( \text{seq}(P,F) \) uniquely determines \( P \).
Local Rules for Growth Diagrams

**Def:** Let $\text{seq}(P, F)$ denote the sequence of partitions along the “staircase”.

\[
\text{seq}(P, F) := (\emptyset, 1, 2, 3, 4, 21, 211, 221, \ldots, 1, \emptyset, \ldots)
\]

**Theorem:** $\text{seq}(P, F)$ uniquely determines $P$. 
Local Rules for Growth Diagrams

**Def:** Let $seq(P, F)$ denote the sequence of partitions along the “staircase”.

$\triangleright\ seq(P, F) := (\emptyset, 1, 2, 3, 2, 3, 4, 3, 31, 21, 211, 221, \ldots, 1, \emptyset)$
Local Rules for Growth Diagrams

**Def:** Let $\text{seq}(P, F)$ denote the sequence of partitions along the “staircase”.

$\triangleright \text{seq}(P, F) := (\emptyset, 1, 2, 3, 2, 3, 4, 3, 31, 21, 211, 221, \ldots, 1, \emptyset)$

**Theorem:** $\text{seq}(P, F)$ uniquely determines $P$. 
Our Reformulation of $\phi^*$
Our Reformulation of $\phi^*$

if $SE \neq NW$

if $SW = NW = SE$

Diagram:

2  21 = SE $\cup$ NW

1  11

11  11 = NW

11  11

11  21 = NW + 1 top row
Our Reformulation of $\phi^*$

if $SE \neq NW$

if $SW = NW = SE$

*if $SW \neq NW = SE$

\[
\begin{array}{c}
\text{2} & \text{21} = SE \cup NW \\
\text{1} & \text{11} \\
\end{array}
\]

\[
\begin{array}{c}
\text{11} & \text{11} = NW \\
\text{11} & \text{11} \\
\end{array}
\]

\[
\begin{array}{c}
\text{11} & \text{11} = NW + 001 \\
\text{1} & \text{11} \\
\end{array}
\]

\[
\begin{array}{c}
\text{11} & \text{21} = NW + 1 \text{ top row} \\
\text{11} & \text{11} \\
\end{array}
\]
Our Reformulation of $\phi^*$

if $SE \neq NW$

if $SW = NW = SE$

*if $SW \neq NW = SE$

Modified Rule for $GDA_k$....

*if last rule makes $|NE| \geq k$ then
Our Reformulation of $\phi^*$
**Our Reformulation of $\phi^*$**

*$GDA_3$ on $(P, F)$*

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Our Reformulation of $\phi^*$

$GDA_3$ on $(P, F)$

$GDA$ on $(\phi^*(P), F)$
Our Reformulation of $\phi^*$

$GDA_3$ on $(P, F)$

$GDA$ on $(\phi^*(P), F)$

Main Theorem: For any rook placement $P$ on a Ferrers board $F$,

$$seq_k(P, F) = seq(\phi^*(P), F)$$
Our Commutation Result

Definition:
Let $P'$ denote the inverse of a placement.

Note:
$\text{seq}(P', F) = \text{rev}(\text{seq}(P, F))$

Corollary:
For any rook placement $P$ on a Ferrers board $F$,
$\phi^*(P') = (\phi^*(P))'$

Proof.
By the Main Theorem and the note above we have:
$\text{seq}(\phi^*(P'), F) = \text{seq}(\text{rev}(\text{seq}(P, F)), F) = \text{rev}(\text{seq}(\phi^*(P), F)) = \text{seq}((\phi^*(P))', F)$

Hence we conclude that $\phi^*(P') = (\phi^*(P))'$.
Our Commutation Result

**Definition:** Let $P'$ denote the inverse of a placement.
Our Commutation Result

**Definition:** Let $P'$ denote the inverse of a placement.

**Note:** $seq(P', F) = rev(seq(P, F))$


Our Commutation Result

**Definition:** Let $P'$ denote the inverse of a placement.

**Note:** $seq(P', F) = rev(seq(P, F))$

**Corollary:** For any rook placement $P$ on a Ferrers board $F$, 

$$\phi^*(P') = (\phi^*(P))'$$
Our Commutation Result

**Definition:** Let \( P' \) denote the inverse of a placement.

**Note:** \( \text{seq}(P', F) = \text{rev}(\text{seq}(P, F)) \)

**Corollary:** For any rook placement \( P \) on a Ferrers board \( F \),

\[
\phi^*(P') = (\phi^*(P))'
\]

**Proof.** By the Main Theorem and the note above we have:

\[
\text{seq}(\phi^*(P'), F) = \text{seq}_k(P', F)
\]
Our Commutation Result

**Definition:** Let $P'$ denote the inverse of a placement.

**Note:** $\text{seq}(P', F) = \text{rev}(\text{seq}(P, F))$

**Corollary:** For any rook placement $P$ on a Ferrers board $F$, 

$$\phi^*(P') = (\phi^*(P))'$$

**Proof.** By the Main Theorem and the note above we have:

$$\text{seq}(\phi^*(P'), F) = \text{seq}_k(P', F)$$

$$= \text{rev}(\text{seq}_k(P, F))$$
Our Commutation Result

**Definition:** Let \( P' \) denote the inverse of a placement.

**Note:** \( \text{seq}(P', F) = \text{rev}(\text{seq}(P, F)) \)

**Corollary:** For any rook placement \( P \) on a Ferrers board \( F \),

\[
\phi^*(P') = (\phi^*(P))'
\]

**Proof.** By the Main Theorem and the note above we have:

\[
\begin{align*}
\text{seq}(\phi^*(P'), F) & = \text{seq}_k(P', F) \\
& = \text{rev}(\text{seq}_k(P, F)) \\
& = \text{rev}(\text{seq}(\phi^*(P), F))
\end{align*}
\]

Hence we conclude that \( \phi^*(P') = (\phi^*(P))' \).
Our Commutation Result

Definition: Let \( P' \) denote the inverse of a placement.

Note: \( \text{seq}(P', F) = \text{rev}(\text{seq}(P, F)) \)

Corollary: For any rook placement \( P \) on a Ferrers board \( F \),

\[
\phi^*(P') = (\phi^*(P))'
\]

Proof. By the Main Theorem and the note above we have:

\[
\text{seq}(\phi^*(P'), F) = \text{seq}_k(P', F) \\
= \text{rev}(\text{seq}_k(P, F)) \\
= \text{rev}(\text{seq}(\phi^*(P), F)) \\
= \text{seq}((\phi^*(P))', F)
\]
Our Commutation Result

**Definition:** Let $P'$ denote the inverse of a placement.

**Note:** $\text{seq}(P', F) = \text{rev}(\text{seq}(P, F))$

**Corollary:** For any rook placement $P$ on a Ferrers board $F$,

$$\phi^*(P') = (\phi^*(P))'$$

**Proof.** By the Main Theorem and the note above we have:

$$\text{seq} (\phi^*(P'), F) = \text{seq}_k(P', F)$$
$$= \text{rev}(\text{seq}_k(P, F))$$
$$= \text{rev}(\text{seq}(\phi^*(P), F))$$
$$= \text{seq}((\phi^*(P))', F)$$

Hence we conclude that $\phi^*(P') = (\phi^*(P))'$. 