On a recent conjecture in pattern avoidance

Jonathan S. Bloom
Rutgers University

AMS Sectional Meeting – Georgetown University, March 2015

Wilf-equivalence

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All patterns τ of length 3 are Wilf-equivalent. Moreover,

$$|\,\mathrm{Av}_n(\tau)| = \frac{1}{1+n}\binom{2n}{n}.$$

Patterns of length 4

We have:

Class | *n*	5	6	7	8	9	...
1 4 2 3	103	512	2740	15485	91245	...
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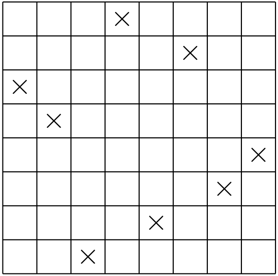
New results

- We give (first) bijective proof that $1\ 4\ 2\ 3 \sim 2\ 4\ 1\ 3$
- Resolves a conjecture of Dokos, et al. (2012)
  - REU group under Sagan

Permutation Statistics

Consider the permutation $\pi = 6\ 5\ 1\ 8\ 2\ 7\ 3\ 4$



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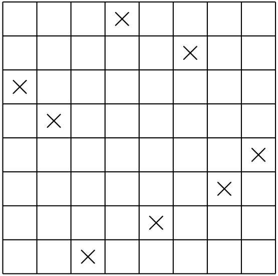

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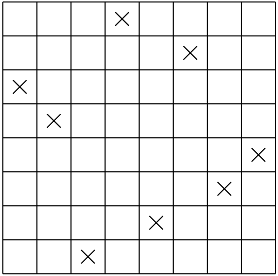

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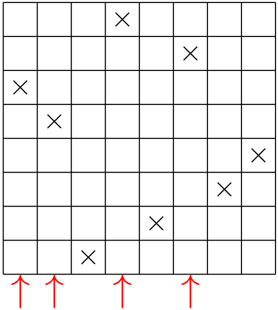

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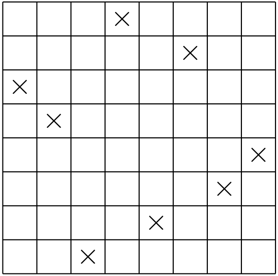

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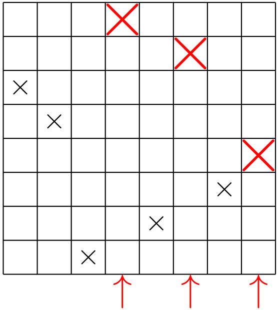

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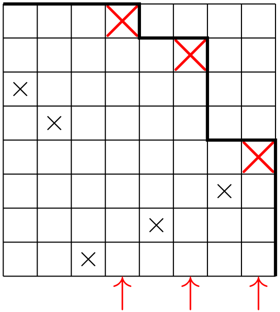

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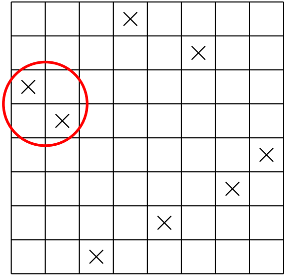

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- $-$ bonds

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Conjecture (Dokos, et al., 2012)

The patterns 1423 *and* 2413 *are Maj-Wilf-equivalent*

- $\mathrm{Maj}(\pi)$ *is sum of descents of* π.

$1423 \sim 2413$ revisited

Theorem (Bloom, 2014)

There is an explicit bijection

$$\Theta : \mathsf{Av}_n(1423) \to \mathsf{Av}_n(2413)$$

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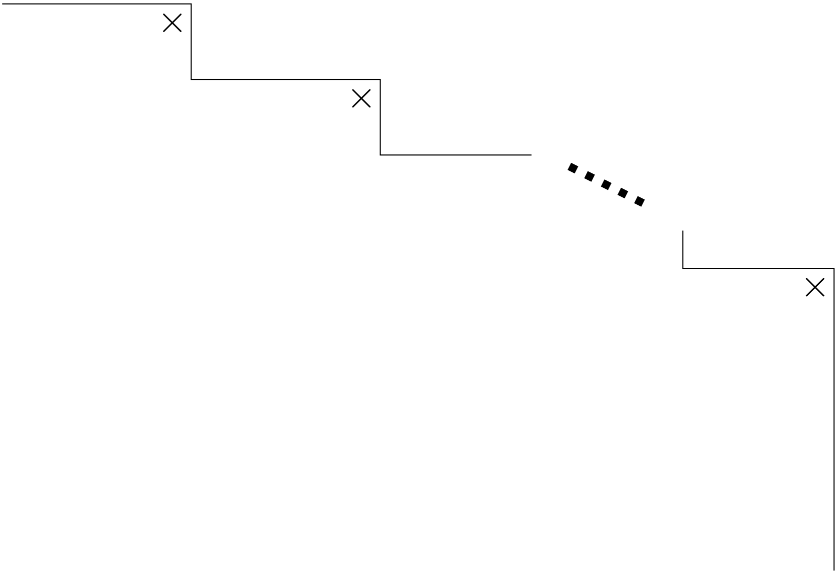
- Stankova's isomorphism does not preserve these statistics.

Anatomy of a 1423



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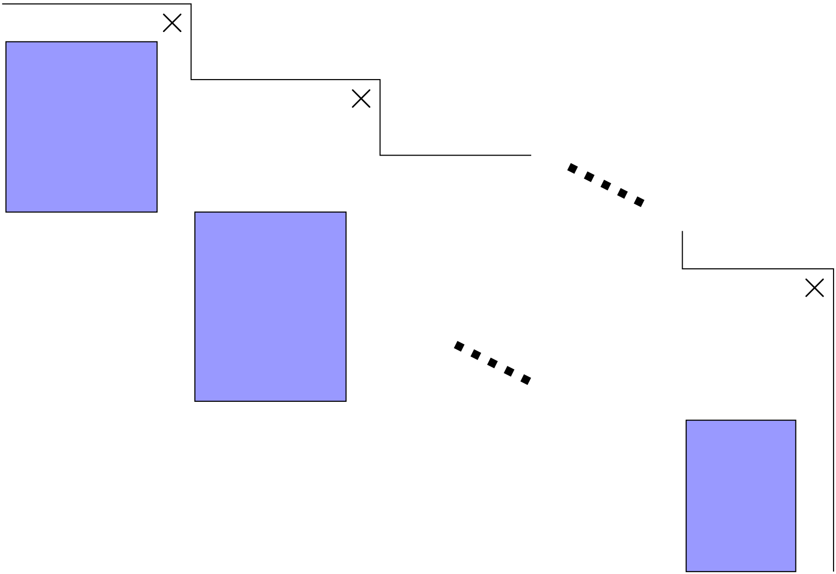

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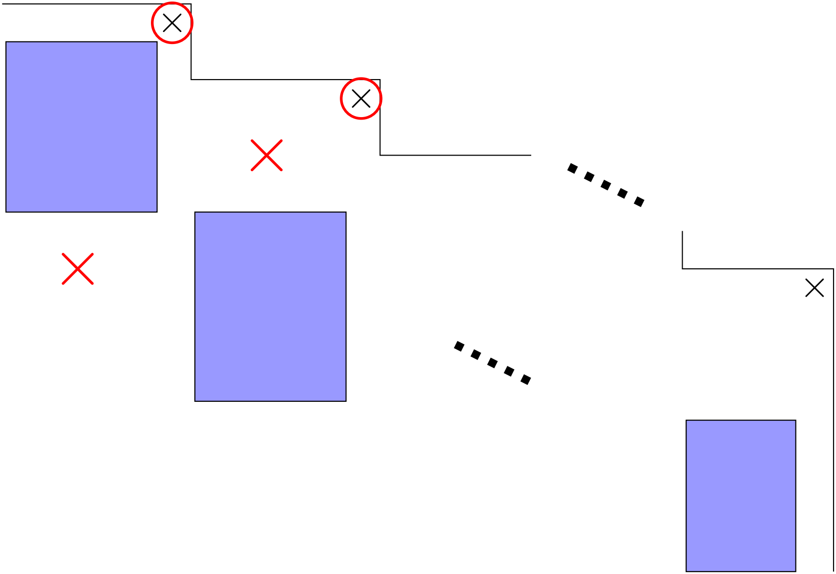
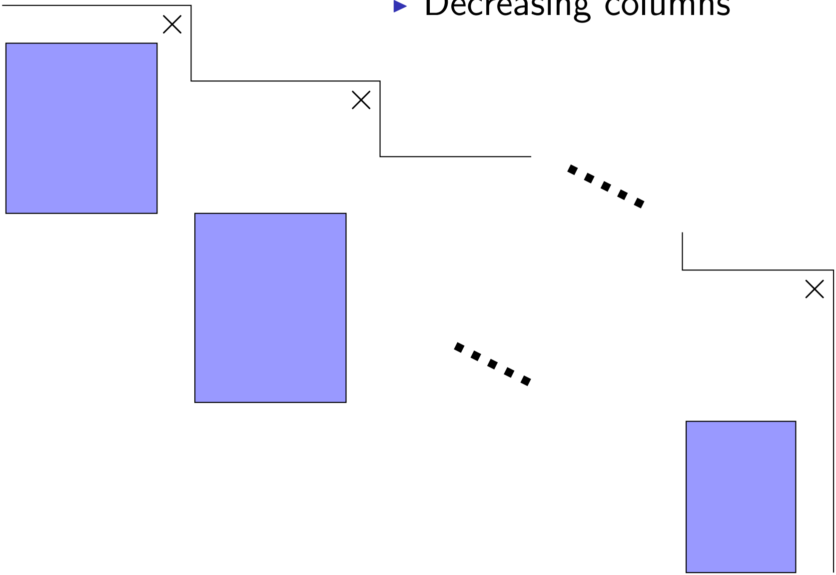
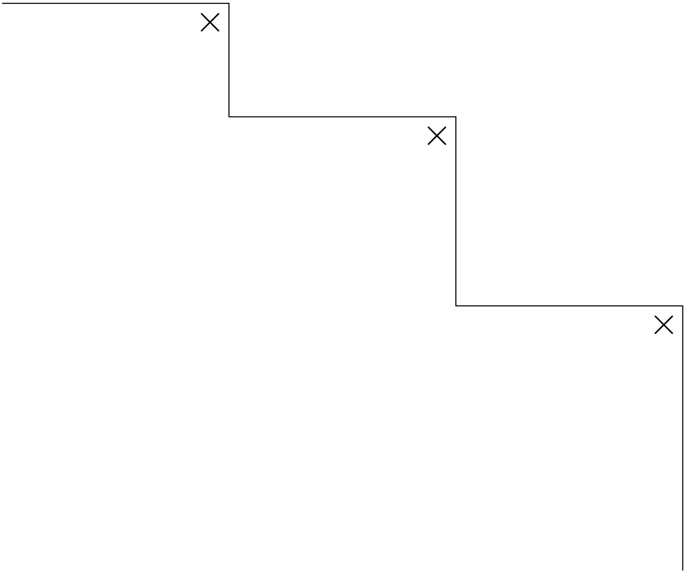


Anatomy of a 1423

- Decreasing columns



Anatomy of a 2413



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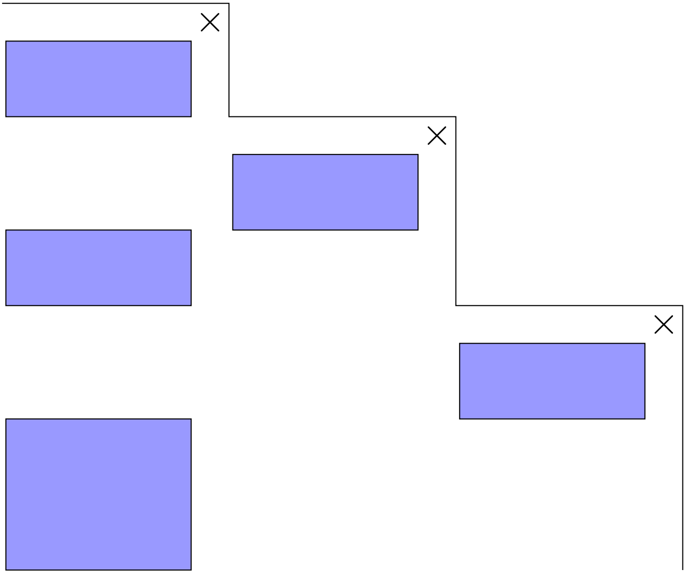

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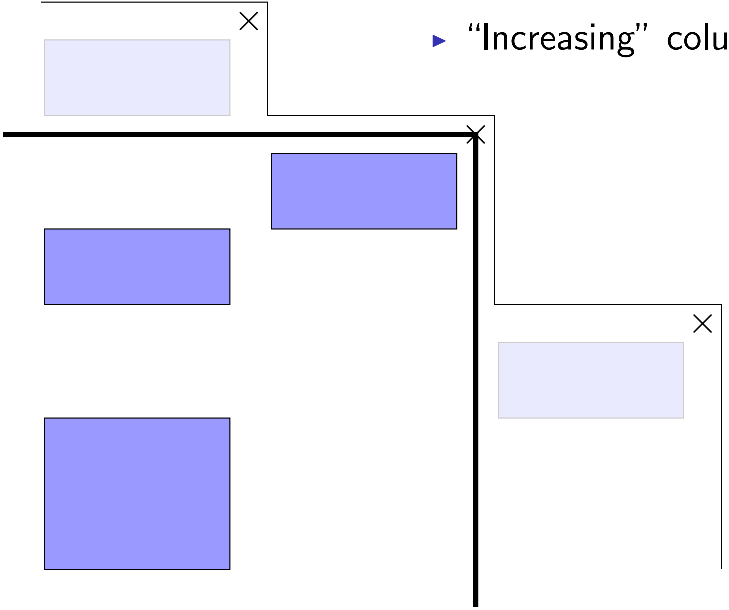

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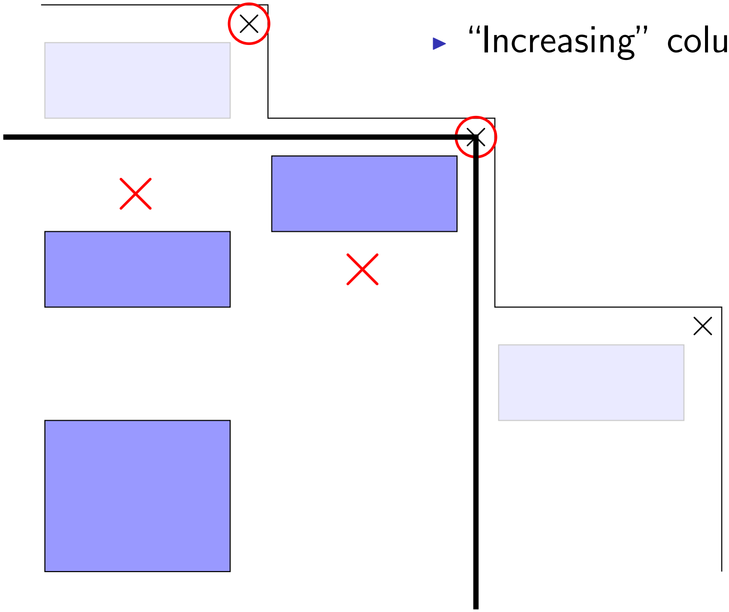

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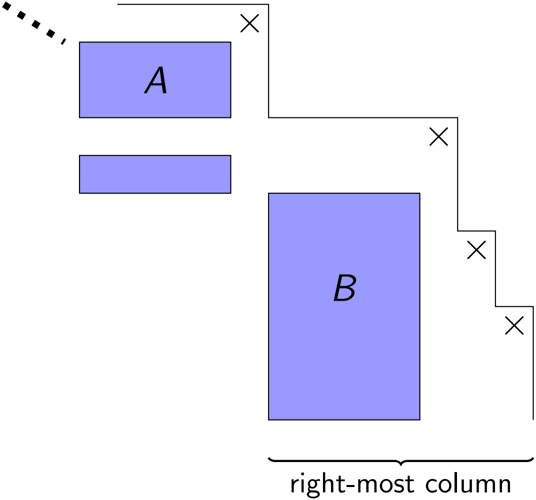
Given $\pi \in \mathrm{Av}_n(1423)$ it decomposes as:



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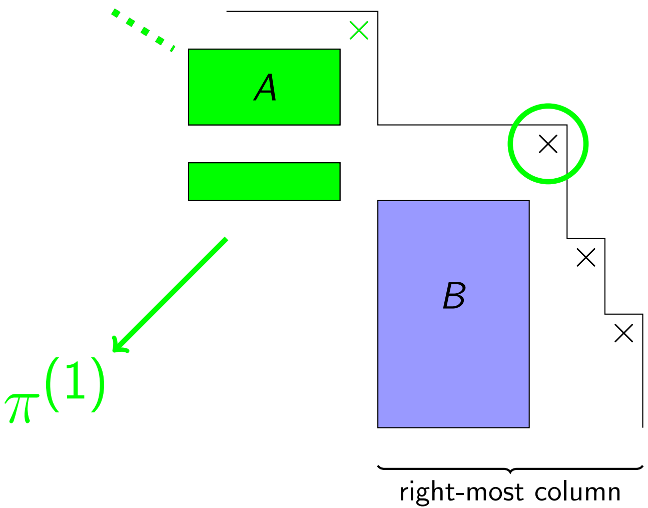

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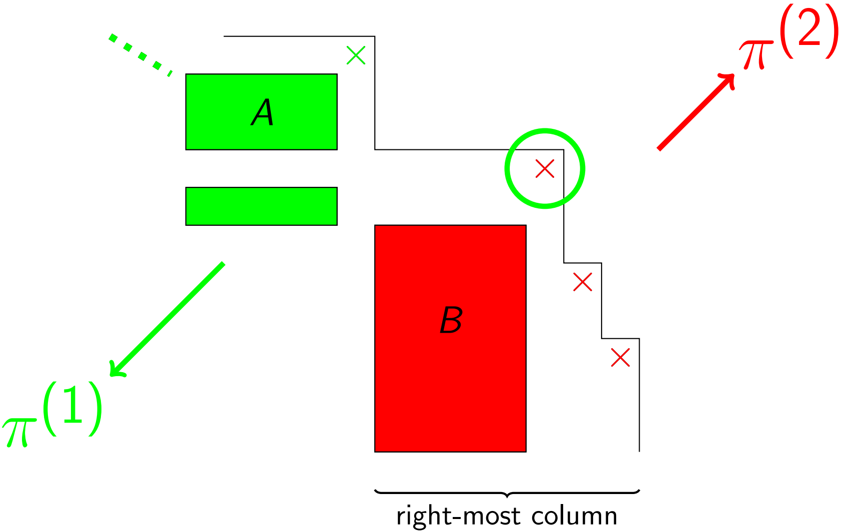
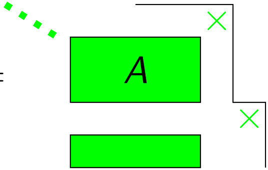
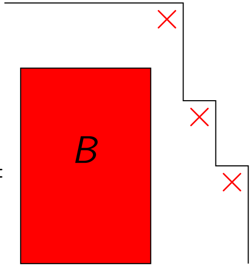
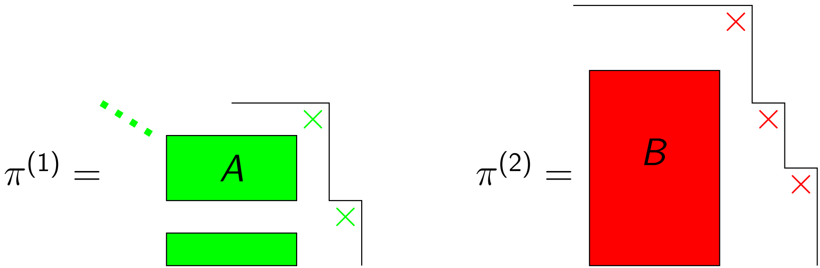


$\pi^{(1)} =$ 

$\pi^{(2)} =$ 



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exists and preserves statistics

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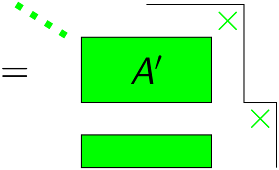

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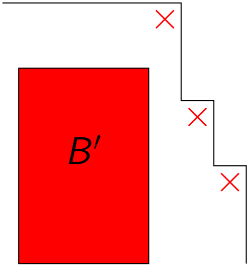

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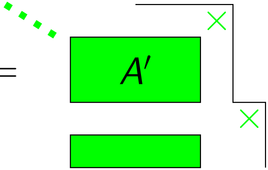

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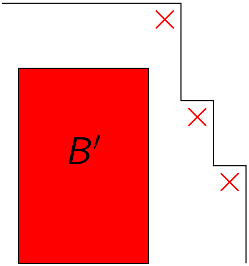

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★ Applying Θ to each part maintains structure!

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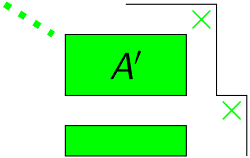

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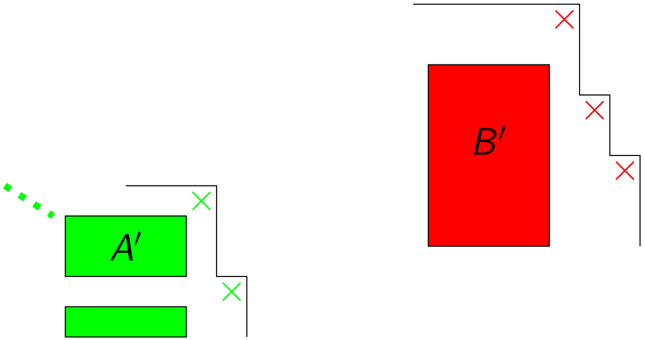

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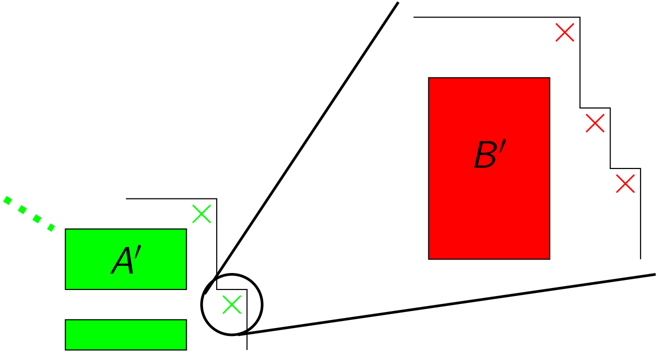
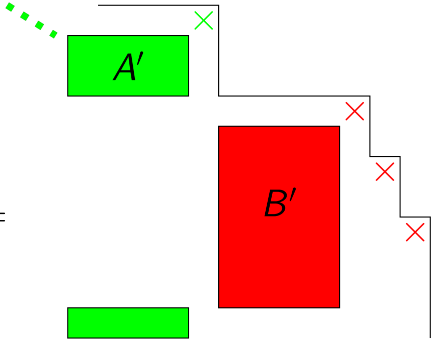


Doing this we obtain our final result:

$$\Theta(\pi) =$$



Egge's Conjecture

Consider the following table

$n =$	2	3	4	5	6	7	...
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Question (Egge):
Are there any patterns $\tau \in S_6$ such that the sets

$$|\,\mathsf{Av}_n(2143, 3142, \tau)|$$

are counted by the large Schröder numbers?

Conjecture (Egge, AMS Fall Eastern Meeting in 2012)

Fix $\tau \in \{246135, 254613, 524361, 546132, 263514\}$. *Then*

$$\sum_{n \geq 0} |\operatorname{Av}_n(2143, 3142, \tau)| x^n = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2},$$

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Proved...

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- *These values of* τ *(and* 180° *rotations) are only patterns*

Proved...

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  - 263514: simple permutations
  - 254613, 524361, 546132: decomposition using LR-maxima
    - Similar flavor (more technical) to $\Theta : \operatorname{Av}(1423) \to \operatorname{Av}(2413)$

Thank You!