

AN ANALYSIS OF THE (COLORED CUBES)³ PUZZLE

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ABSTRACT. Start with a collection of cubes and a palette of six colors. We paint the cubes so that each cube face is one color, and all six colors appear on every cube. Take n^3 cubes colored in this manner. When is it possible to assemble these cubes into an $n \times n \times n$ large cube so that each face on the large cube is one color, and all six colors appear on the cube faces? For the $2 \times 2 \times 2$ case we give necessary and sufficient conditions for a set of eight cubes to have a solution. Furthermore, we show that the (colored cubes)³ puzzle always has a solution for $n > 2$.

1. INTRODUCTION

There are many problems in mathematics that are inspired by games and mathematical recreations. This paper is the result of our investigation into a relative of a solitaire game known as Instant Insanity¹. This popular game, marketed in the late 60's by Parker Brothers, consists of four cubes, each of which is colored in some way with four colors. The goal of Instant Insanity is to stack the four cubes into a $4 \times 1 \times 1$ prism so that each of the four colors appears on each 4×1 face of the prism. Although tame when compared to today's world of complicated electronic gadgets and realistic video games, Instant Insanity still remains an interesting diversion for people who like to get their hands on a puzzle for a change.

The Instant Insanity puzzle has the added (mathematical) advantage that it can be solved using an elegant and elementary application of graph theory. Treatments of the solution can be found in classic references, such as Chartrand's *Graphs as Mathematical Models* [1], or in more contemporary textbooks like Tucker's *Applied Combinatorics* [6] and van Lint and Wilson's *A Course in Combinatorics* [7]. However, this puzzle also has unexpected depth; Robertson and Munro were able to show that a generalization of the game, with n cubes and n colors, is an NP-complete problem [5]. Another version of Instant Insanity with different Platonic solids was also studied by Jebasingh and Simoson [2].

The version of the game we consider, the (colored cubes)³ puzzle, uses (as the name suggests) many more cubes. Given n^3 cubes, the puzzle is solved if the cubes can be stacked into an $n \times n \times n$ cube so that each $n \times n$ face is colored a single color. This can be a considerable challenge. Since each cube can be placed in any one of 24 orientations, even the $2 \times 2 \times 2$ puzzle has $\frac{8!(24)^8}{24} \cong 1.85 \times 10^{14}$ possible configurations. Perhaps surprisingly, the $2 \times 2 \times 2$ case is the only one which does not always have a solution. Since $(n-2)^3$ cubes are "invisible" and another $6(n-2)^2$ contribute only one face to the solved puzzle, the puzzle becomes easier to solve as n increases. We introduce the *frame* of the puzzle, consisting of the cubes on the

1991 *Mathematics Subject Classification.* 52C99.

This research was supported by NSF grant DMS-0243763.

¹Instant Insanity is a trademark of Parker Brothers.

corners and edges of the puzzle. The frame contains the $n^3 - (n-2)^3 - 6(n-2)^2 = 12n - 16$ cubes that must be placed in a non-trivial way in a solution. As the number of cubes available to fill the frame is n^3 , this suggests that the (colored cubes)³ puzzle should become easier to solve as the value of n increases.

Our main theorem is that the (colored cubes)³ puzzle always has a solution for $n \geq 3$. The proof is in two parts: we first show that when we can find a $2 \times 2 \times 2$ solution of eight cubes out of the n^3 for the corners of the puzzle, then we can always complete the puzzle's frame when $n \geq 3$. Next, we show that as long as there are 27 cubes, we can always find a $2 \times 2 \times 2$ solution. Along the way, we construct a set of 22 cubes with no subset forming a $2 \times 2 \times 2$ solution. As every $n \times n \times n$ puzzle contains a $2 \times 2 \times 2$ sub-puzzle at its corners, this result suggests that a solution to a $3 \times 3 \times 3$ puzzle, at least, might be quite hard to find. Indeed, showing that the case $n = 3$ has a solution is the most involved result in this paper.

We remark that our version of this problem is not entirely new. A variation of the problem was asked as early as 1921 by Percy Alexander MacMahon, a major in the British Army, in [4]. Major MacMahon started with a collection of the 30 possible distinct cubes colored with six colors and constructed particularly nice $2 \times 2 \times 2$ solutions. Other similar problems can be found at a web site run by Jürgen Köller [3].

We lay out this paper as follows: In the next section, we give background to the (colored cubes)³ puzzle and introduce the definitions and terminology we use throughout the paper. In section 3, we describe a necessary and sufficient condition for eight cubes to have a solution. We then show that the solution to our puzzle for $n > 2$ is completely dependant on a solution for the corner triples. In section 4, we show how to construct a corner solution when $n = 3$. This also implies a solution for $n > 3$. In the final section, we describe how the puzzle can be generalized, and give some open questions. We hope to convince the reader that the (colored cubes)³ puzzle, analyzable with standard combinatorial tools, still has a lot of room for further investigation.

2. BASIC NOTIONS AND BACKGROUND

Definition 2.1. *A colored cube is a cube whose faces each have a distinct color.*

In this paper we will always assume that there are only six colors, so every color appears on every colored cube. One cube is considered equivalent to another if it is possible to apply a rigid motion in \mathbb{R}^3 to the first cube so that its face colors coincide with the second. In particular, two cubes related by a mirror operation are not equivalent.

Definition 2.2. *The equivalence class of cubes under the preceding equivalence relation is called the variety of colored cube.*

When it is clear from context, we will occasionally refer to both a cube and its variety as c . Under our assumptions there are 30 varieties of colored cubes. This is not hard to see: pick some distinguished color to be the front face. This leaves five choices for the back face, and $3!$ choices for the remaining four faces. We next define the problem we solve in this paper.

Definition 2.3. *A (colored cubes)³ puzzle (or $n \times n \times n$ puzzle) is an arbitrary set of n^3 colored cubes of the same size. These colored cubes can be of any variety.*

A solution to an $n \times n \times n$ puzzle is an arrangement of the colored cubes into an $n \times n \times n$ cube where each $n \times n$ face is a solid color and all six colors appear on the outside of the cube.

We note that the $n \times n \times n$ cube created in an $n \times n \times n$ solution is a scaled-up version by a factor of n of some $1 \times 1 \times 1$ colored cube. Thus, it is natural to define the following:

Definition 2.4. A solution cube is said to be modeled after a colored cube if its faces have the same coloring, up to rigid motion, as the colored cube. In other words, the solution and its model are of the same variety.

A number of the arguments in this paper require information about the relative position of cube faces. The next definitions identify the most important relationships.

Definition 2.5. An opposite pair is an unordered pair of colors taken from opposite faces of a colored cube. Similarly, an adjacent pair is an unordered pair of colors taken from adjacent faces of a colored cube. A corner triple is an ordered triple of colors which correspond to three faces taken clockwise about a corner of a colored cube.

Two corner triples are *equivalent* when one is a rotation of the other. For example, $(1, 2, 4) \sim (2, 4, 1) \not\sim (1, 4, 2)$. We say two corner triples are *mirror image* corner triples if they contain the same colors but are not equivalent. Using these definitions, we see that a colored cube has three opposite pairs, twelve distinct adjacent pairs corresponding to its edges, and eight corner triples corresponding to its corners. Since all six colors appear on any cube, we immediately see that any two face colors are present in a colored cube either as an adjacent pair or as an opposite pair.

We also identify a useful construction. Let c^* , the *mirror image cube* of c , be the colored cube obtained by central inversion on the cube c . (That is, swap the colors on all opposite faces.) We can show that when two cubes share the same opposite faces, they are either identical or mirror images of each other. Consider two cubes that have the same opposite face pairs. Arrange the cubes so the same color face is forward. Then the rear face colors also match. We next rotate the cubes about the line through the center of the front and back faces so the same colors appear on the left (and right) faces. There are now only two possibilities: either the top and bottom faces match or they have opposite colors. In the former case, the two cubes are of the same variety; in the latter, they are mirror images. This gives us the characterization of c^* as the cube c with any opposite face pair swapped. We also note that the corner triples of c^* are mirror images of the corner triples of c .

When we compare cubes, we are usually interested in what components the two cubes have in common. The next two lemmas show how the opposite pairs on two cubes determine common adjacent pairs and corner triples.

Lemma 2.6. Two colored cubes share exactly nine, ten, or twelve adjacent pairs according to whether they share exactly zero, one, or three opposite pairs, respectively.

Proof. There are $\binom{6}{2} = 15$ unordered pairs of distinct colors, all of which appear in every colored cube as adjacent or opposite pairs. Three of the pairs are opposite

pairs while the rest are adjacent pairs. Colored cubes c_1 and c_2 may share zero, one, or three opposite pairs, thereby giving six, five, or three distinct opposite pairs. Subtracting these from the 15 total pairs yields the result. \square

Lemma 2.7. *Given two cubes:*

- (1) *If they have no opposite pairs in common, then they share zero or two corner triples.*
- (2) *If they have one opposite pair in common, then they share exactly two corner triples.*
- (3) *If they have three opposite pairs in common, then they share zero or eight corner triples.*

Proof. Given a variety c , we can describe the relation between c and the other 29 varieties. Once we understand these relationships the results will follow. Some of the 29 varieties can be constructed from c by swapping the colors along some edge of c . A cube has twelve edges, so this accounts for twelve varieties. Other varieties can be constructed by choosing a corner triple of c , then cyclicly permuting the colors on that corner triple. If one performs this process on corner pairs (fix a corner triple and permute the corner triple opposite on the long diagonal of the cube), three varieties, c and two others, are generated. Therefore, eight varieties besides c (two for each corner pair) can be built in this way. In a similar manner, there are another eight varieties that are the mirror images of the ones generated by a corner permutation. The final variety of the 29 is c^* .

We note that 29 varieties generated from c are distinct from c . The twelve varieties generated from by swapping adjacent faces of c share one opposite pair with c , the opposite pair that remains unchanged after the swap. Each of these twelve varieties also shares two adjacent corner triples with c . These are the corner triples formed by the four faces whose colors remain fixed. The eight varieties generated by cyclicly permuting a corner triple of c share no opposite pairs with c . However, these varieties do share corner triples, the permuted corner triple and its opposite corner. The remaining six corner triples cannot be corner triples of c as some two of their three colors are opposite in c . Similarly, the eight varieties in the mirror image set also share no opposite pairs with c , as mirror images preserve the color pairing. Furthermore, these cubes share no corner triples with c . Six of the corner triples have the wrong colors, while the last two corner triples have the wrong orientation. The final variety, c^* , shares three opposite pairs with c but no corner triples; the colors are right, but their orientation is not.

To complete the proof, we show that the 29 varieties are also distinct from each other. This can be most easily seen by comparing which corner triples each variety has in common with c . Each of the twelve face-swapped varieties share two unique adjacent corner triples with c . Two each of the eight corner-twisted varieties share two corner pairs with c and these two are clearly different from each other. This implies that the eight mirrored varieties are also distinct from each other, and as they share no corner triples with c they are also distinct from the previous two subsets. Finally, c^* cannot be in any of the previous subsets as it is the only variety of the 29 with three opposite pairs in common with c . \square

3. SOLVABILITY OF AN $n \times n \times n$ PUZZLE

The first non-trivial (colored cubes)³ puzzle is the case $n = 2$. It is easy to construct $2 \times 2 \times 2$ puzzles both with and without solutions. For example, take eight cubes of some variety c . Clearly, they can be stacked into a $2 \times 2 \times 2$ solution modeled after c . On the other hand, take seven cubes of variety c and one of variety c^* . Since c^* shares no corner triples with c , it cannot be used in a solution modeled after c . There cannot be a solution modeled after any other cube either. By Lemma 2.7, c shares at most two corner triples with any other cube, so at most two copies of c can be used in a solution not modeled after c .

The following theorem gives necessary and sufficient conditions for a $2 \times 2 \times 2$ puzzle to have a solution.

Theorem 3.1. *Let K be a set of eight corner triples from eight distinct cubes. The corner triples of K can be arranged into a $2 \times 2 \times 2$ solution if and only if the following three conditions are satisfied:*

- (1) *Each color appears as a face on exactly four elements of K .*
- (2) *No two elements of K are mirror image corner triples.*
- (3) *When the corner triples in K are decomposed into adjacent pairs, each adjacent pair (a, b) occurs twice in the list, once coming from a corner triple (a, b, c) and once from (b, a, d) .*

Proof. By considering an arbitrary $2 \times 2 \times 2$ colored cube, it is easy to verify that these conditions are necessary. For sufficiency, we need a way to keep track of eight corner triples and their different faces. We number the eight component corner triples as in Figure 3.1 which shows the top view of a “smashed cube.” Corner triples 1–4 are on the top, listed in counterclockwise order. Right below them are corner triples 5–8. In order to determine how the faces of the corner triples interrelate, we use Figure 3.2, which shows the standard net of a cube. Numbers in the subsquares refer to the corner triples in Figure 3.1. The letters refer to faces of the $2 \times 2 \times 2$ cube; U for “up,” L for “left,” D for “down,” R for “right,” B for “back,” and F for “front.” For example, there are squares labeled 1U, 1B, and 1R. This means that corner triple 1 contributes a face on the “up,” “back,” and “right” faces of the $2 \times 2 \times 2$ cube. We also note that in a $2 \times 2 \times 2$ solution, we can think of the letters as distinct colors.

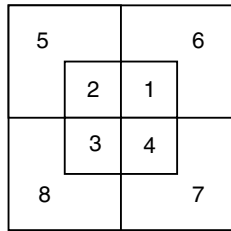


FIGURE 3.1.
Corner Triple
Positions

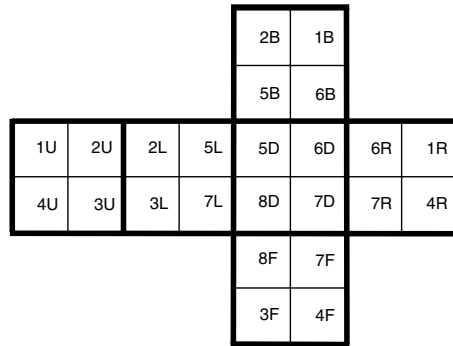


FIGURE 3.2.
Cube Faces

Assume that we have a collection of eight corner triples as in the statement of the Theorem. Let any one of the corner triples, appropriately placed, be corner triple 1, and let U, B, and R be the triple's three colors. By condition 3, there is exactly one other corner triple with the colors UB. We place this triple in position 2 so that the colors match with those on triple 1. The color of 2L must be distinct from 1R as no two corner triples are mirror images by condition 2. Using an identical argument we can place the corner triple containing colors UR into position 4 with the colors 4F and 1B distinct. Further, we claim that colors 2L and 4F are also distinct. For if they are the same, we have found the two corner triples with adjacent faces with colors U and F. By condition 1, there is one more corner triple with a face color U. By condition 3, when that corner triple is decomposed, it must generate two pairs UC and CU, where C is some color. This implies that the corner triple's colors are UCC, which contradicts our definition of colored cube. Since F and L are distinct colors, there is one more corner triple with U as a face, and with UF and UL as adjacent pairs. By condition 3, it has the appropriate orientation to fit into condition 3. This completes the top layer.

By condition 3, there is exactly one other corner triple with colors LB, which we place in position 5. By the mirror image condition 2, the third color on corner triple 5, color D, is distinct from U. We can now place corner triples 6 and 7 appropriately by condition 1 (to identify the corner triple), and condition 3 (to insure that the remaining colors on the corner triples match the colors that have already been placed. The last corner triple fits into position 8 by condition 3. \square

One can use the criteria in Theorem 3.1 to obtain an empirical estimate of the frequency of solutions in the $2 \times 2 \times 2$ puzzle; by checking through a few thousand random cases we found a $2 \times 2 \times 2$ solution just over 50% of the time.

There are also other situations when we can say something about the existence of a $2 \times 2 \times 2$ solution. For example,

Lemma 3.2. *Given any set of 15 colored cubes, all of which share one opposite pair, then there is a subset of eight cubes that forms a $2 \times 2 \times 2$ solution.*

Proof. Without loss of generality, let white and black be the colors in the distinguished opposite pair. We note that there are a total of six different cube varieties with white and black faces opposite. The six varieties can be further split into three pairs, each consisting of a variety and its mirror image. From Lemma 2.7, we know that two distinct cube varieties with the same opposite pair are either mirror images of each other or differ by a swapped adjacent pair. Two varieties from the set of six share corner triples if and only if they differ by a swapped adjacent pair, and the two matching corner triples are the ones formed by the four faces whose colors agree. In particular, only a variety and its mirror image have no corner pairs in common.

Given an arbitrary set of cubes with an opposite face in common, let c be a cube with the highest repetition number. If cube c occurs seven times, then any other cube besides c^* shares two corner triples with c by Lemma 2.7. That cube and the seven copies of c can be assembled into a $2 \times 2 \times 2$ solution modeled after c . Therefore, the largest incomplete set consists of seven each of c and c^* . At the other extreme, assume that the cube c occurs only twice. We note that once there are four cubes of the six that occur twice then there is a solution. Consider a cube c' from the six varieties whose mirror image does not occur twice. It will share

two corners with each of the four cubes that occur twice. Since the four pairs of corners are distinct, they can be used to form a $2 \times 2 \times 2$ solution modeled after c' . Similarly, if three of the six varieties occur twice and the other three occur only once then there is also a $2 \times 2 \times 2$ solution. There are nine cubes total, and there will be a solution modeled after the cube whose mirror image only occurs once. On the other hand, the reader can check that there are collections of eight cubes where no variety occurs more than twice without a $2 \times 2 \times 2$ solution.

Using analogous arguments, we can determine that the maximal sizes of incomplete sets where cube c occurs $(6, 5, 4, 3)$ times is $(13, 13, 12, 12)$. The result follows. \square

The $2 \times 2 \times 2$ solution is particularly important because every solution to a $n \times n \times n$ puzzle starts with a $2 \times 2 \times 2$ solution in its eight corner cubes. Once the corner solution is found, we need to find cubes to fill the remaining open edge positions. The solution requires twelve distinct subsets of $n-2$ cubes, one subset for each edge. The cubes in each subset must all have the appropriate adjacent pair for that edge. If these can be found, then the puzzle is solved. The remaining cubes that are visible have only one face showing, and since all colors appear on each cube, any cube oriented in the proper manner can fill those places. The orientation of the cubes that make up the $(n-2) \times (n-2) \times (n-2)$ interior does not matter. These observations motivate the following definition:

Definition 3.3. *The frame of an $n \times n \times n$ puzzle consists of the eight corner and $12(n-2)$ edge cubes of the puzzle.*

A puzzle is solved once its frame is solved. Since the number of puzzle cubes grows as n^3 while the size of the frame grows as n , the $n \times n \times n$ puzzle should become easier to solve as n increases. However, there is still some work to do to show when there is a guaranteed solution to the frame. The next two results show that once a $2 \times 2 \times 2$ solution has been found, it is always possible to solve the frame (although not necessarily in a way that uses the $2 \times 2 \times 2$ solution).

Theorem 3.4. *A $3 \times 3 \times 3$ puzzle has a solution if it contains a subset with a $2 \times 2 \times 2$ solution.*

Proof. Once a $2 \times 2 \times 2$ solution is found, it is sufficient to complete the frame of the cube in order to finish the $3 \times 3 \times 3$ puzzle. In other words, if twelve of the remaining 19 cubes can be oriented and placed along the edges in such a way that all the colors line up, then there is a solution to the $3 \times 3 \times 3$ puzzle. We assume that this is not the case, so that there is no possible way to complete the frame given the initial $2 \times 2 \times 2$ solution. We will show that eventually this situation always yields another $2 \times 2 \times 2$ solution for the corner triples that can be completed.

Start with a $2 \times 2 \times 2$ solution for which there is no corresponding edge solution. Label the 12 adjacent pairs as e_1, e_2, \dots, e_{12} . Let c_i be the number of cubes, out of the 19, in which e_i appears. Without loss of generality, we may assume the adjacent pairs are labeled so that the sequence $\{c_i\}$ is in ascending order.

We note that if $c_i \geq i$, then there is always a set of twelve cubes which can be used as a matching for the twelve adjacent pairs needed for the frame's solution. If there is no solution, then there must be some largest j with $c_j < j$. In a worst case, for every $i \leq j$, $c_i = j - 1$, and for every $i > j$, $c_i = 19$. In other words, we

have the inequality

$$\sum_{i=1}^{12} c_i \leq (j)(j-1) + (12-j)(19).$$

On the other hand, $19 \times 9 \leq \sum_{i=1}^{12} c_i$, as two cubes always have at least nine adjacent pairs in common by Lemma 2.6. If we combine the inequalities and solve for j over the integers, we find that $j \leq 3$ or $j \geq 17$. The latter case cannot occur as $j \leq 12$. Therefore, if there is a solution for the corner triples that cannot be completed to a solution for the frame, it occurs because there are at most three adjacent pairs with insufficient representatives. We will cover this situation in cases:

Case 1: Assume that $c_1 = 0$. This means that there is an adjacent pair that does not occur in the collection of cubes. As a result, the two colors making up the adjacent pair, say black and white, are opposite in these 19 cubes. Therefore, by Lemma 3.2 there must be a second $2 \times 2 \times 2$ solution for the corner triples with black and white opposite. We still need to show that we can find adjacent pairs to complete the solution. Label the twelve adjacent pairs in the corner solution and their number of occurrences as above. Of the 19 cubes, 11 are not used in the second $2 \times 2 \times 2$ solution. As black and white are opposite on these cubes, they have at least ten adjacent pairs in common with the $2 \times 2 \times 2$ solution by Lemma 2.7. We also have the 8 cubes from the original $2 \times 2 \times 2$ solution, which may share as few as nine adjacent pairs with the new $2 \times 2 \times 2$ solution. We get another inequality,

$$8 \times 9 + 11 \times 10 \leq \sum_{i=1}^{12} c_i \leq (j)(j-1) + (12-j)(19),$$

which has a solution over the integers when $j \leq 2$. The only ways now that the frame can fail to have a solution is if either $c_1 = 0$ or if $c_1 = c_2 = 1$ and contributions to e_0 and e_1 come from the same cube.

For the second $2 \times 2 \times 2$ solution, assume again that $c_1 = 0$. Then the second collection of 19 cubes lacks the same adjacent pair, and that pair must be opposite on the 19 cubes. Note also that the adjacent pair cannot be black and white as the second $2 \times 2 \times 2$ solution has black and white opposite. Therefore, 11 of the 19 cubes actually share three opposite pairs, so they have the same adjacent pairs by Lemma 2.6 and at least six of them are identical. Assume, for a moment, that of the remaining eight cubes out of the 19, at least two share corners with the six identical cubes. Use these to build a *third* $2 \times 2 \times 2$ solution. Use the eight cubes from the second $2 \times 2 \times 2$ solution and any other cube to fill in nine edges of the frame. There are still four cubes remaining that have the same opposite pairs as the corner solution, which guarantee that the frame can be completed.

What if no cubes share two corners with the third $2 \times 2 \times 2$ solution? By Lemma 2.7 this can only happen if those cubes are the mirror image of the $2 \times 2 \times 2$ solution. In this case, it is straightforward to pick a corner solution and complete the frame.

Finally, we note that if $c_1 = c_2 = 1$ and contributions to e_0 and e_1 come from the same cube, then the 18 remaining cubes lack the same two adjacent pairs. Therefore, these pairs must be opposite and the 18 cubes have the same three opposite faces. There are only two such cubes, so the pigeonhole principle guarantees that at least nine are identical, and the rest of this case follows as above.

Case 2: Assume that $c_2 = 1$. Then we can assume that $c_1 = 1$ and that the contributions to e_1 and e_2 come from the same cube. This situation was covered in the previous case.

Case 3: Assume that $c_3 = 2$. Further, we can assume that $c_1 = c_2 = 2$. The only way that the frame can fail to have a solution is if e_1 , e_2 , and e_3 occur on the same two cubes. In this situation there are 17 cubes that have the same three opposite faces. There are only two such cubes, at least nine are identical, and the rest of this case follows as before. \square

Theorem 3.5. *An $n \times n \times n$ puzzle with $n > 3$ has a solution if it contains a subset with a $2 \times 2 \times 2$ solution.*

Proof. The argument is analogous to the proof of Theorem 3.4. Assume that there is a $2 \times 2 \times 2$ corner solution that cannot be completed to a solution of the $n \times n \times n$ frame using the remaining $n^3 - 8$ cubes. Define, as before, e_i and c_i , although this time c_i is the number of cubes, out of the remaining $n^3 - 8$, in which the pair e_i appears. If necessary, reorder the adjacent pairs so the corresponding c_i are arranged in increasing order. A sufficient condition for a solution is $c_i \geq (n - 2)i$ for all i , so denote by j the last place where this condition fails. We then have the following inequalities:

$$(n^3 - 8) \times 9 \leq \sum_{i=1}^{12} c_i \leq (j)((n - 2)j - 1) + (12 - j)(n^3 - 8).$$

The outside inequalities can be converted to an equality which is quadratic in j . It has solutions

$$j = \frac{-7 + n^3 \pm \sqrt{-143 + 96n + 10n^3 - 12n^4 + n^6}}{-4 + 2n}.$$

The larger solution grows without bound, and is always greater than 12 for $n \geq 3$, whereas the smaller solution decreases to 3 as n increases, and is always smaller than 4 for $n \geq 3$. As a result, $j = 3$ is the largest value for which it is possible for $c_j < (n - 2)j$ to hold. We conclude that when the $n \times n \times n$ frame cannot be completed from a $2 \times 2 \times 2$ corner solution, there is some adjacent pair which appears no more than $3(n - 2) - 1 = 3n - 7$ times. The remaining $n^3 - (3n - 7) - 8 = n^3 - 3n - 1$ cubes have these colors on opposite faces.

There are only six varieties which share one pair of opposite faces. Partition this set into three subsets, each containing cubes of a variety and its mirror image. As $n^3 - 3n - 1 > 45$ is always true for $n > 3$, there is one subset that contains at least 15 members, which in turn implies that there are at least eight cubes of one variety, say c . The eight copies of c become the new $2 \times 2 \times 2$ corner cubes. At least $9(n - 2)$ of the remaining edge positions in the puzzle can be filled in at random, as any cube shares at least nine adjacent pairs with the solution cube. Fill these edge positions using cubes of any variety except c and c^* . To finish the argument, we note that there are at most $3(n - 2) = 3n - 6$ adjacent pairs left to fill to complete the frame. By Lemma 2.7, these pairs all appear on both c and c^* . As $\frac{n^3 - 3n - 1}{3} > 3n + 2 = (3n - 6) + 8$ when $n > 3$, there are enough copies of c and c^* to fill in the outstanding edge positions as well as the corners. \square

Corollary 3.6. *A $n \times n \times n$ puzzle always has a solution for $n \geq 6$.*

Proof. By the extended pigeonhole principle, we need $7 \times 30 + 1 = 211$ cubes in order to guarantee at least eight colored cubes of one variety. This will always happen when $n \geq 6$. \square

4. A SOLUTION TO THE $3 \times 3 \times 3$ PUZZLE

In the previous section, we showed that when a (colored cubes)³ puzzle has a $2 \times 2 \times 2$ solution for the corners, the rest of the puzzle can be completed. In this section, we show that every collection of 27 cubes contains a $2 \times 2 \times 2$ solution. It would be desirable to have a succinct proof of this fact, but so far this has eluded us. Instead, we were able to write some *Mathematica* code to check a number of basic cases. This computational lemma provides enough structure to finish the result.

Lemma 4.1. *Given the following sets of cubes, one can always find a subset of eight that forms a $2 \times 2 \times 2$ solution. (A k -tuple refers to the number of times each cube variety appears in the set.)*

- seven distinct doubles.
- six distinct triples.
- five distinct 4-tuples
- four distinct 6-tuples.
- three distinct doubles and two distinct 4-tuples.
- two distinct doubles and two distinct 6-tuples.

Proof. This lemma was verified case-by-case using an exhaustive search in *Mathematica*. \square

We should say a bit about the way the *Mathematica* search was performed. We used a 30×40 matrix of 0's and 1's as the data structure; rows corresponded to the 30 cube varieties and columns corresponded to the 40 different corner triples. The (i, j) th entry in the matrix was non-zero when the i th cube contained the j th corner triple. The algorithm looked at all distinct k -subsets of varieties of cubes and determined the corner set—the set of corner triples that were represented by this subset. Then the algorithm determined if the corner set contained eight corner triples from any of the 30 possible cubes. (The check was slightly different depending on whether the cube being considered was a member of the subset or not.) It was important in our algorithm to assume that each distinct variety in the subset had a least two representatives. Since two varieties share up to two corner triples, this assumption guaranteed that both shared corners would be filled by a given variety. A variation of our algorithm could be developed for subsets that include varieties with a single representative cube, but the modified code would need to confirm that in a potential solution each corner triple could be represented by a different cube. The first author has posted some of the *Mathematica* code used in this paper on his web page for the interested reader.

We build on the cases listed in Lemma 4.1 to show that any arbitrary set of 27 cubes has a subset of eight cubes with a $2 \times 2 \times 2$ solution. We will need some new notation for the results that follow. We start with a notion of the most common variety of cube in a set that we alluded to in Lemma 3.2. We make this a bit more formal here.

Definition 4.2. *Let $r(c_i)$, the repetition number of the variety c_i , be the number of cubes of variety c_i in the puzzle.*

The cube variety with the greatest repetition number will figure prominently in the rest of this paper, and we will refer to this variety as c_M . It may happen that more than one cube variety has the same maximal repetition number. In this case, arbitrarily pick one variety.

Definition 4.3. Define C to be the set containing c_M , c_M^* , and the eight varieties that share no corner triples with c_M . Define C' to be the complementary set to C in the set of the 30 possible varieties. Given a set of puzzle cubes, we let V_C be the number of varieties in the puzzle that are in the set C . Similarly, we let $V_{C'}$ be the number of varieties in the puzzle that are in the set C' .

By Lemma 2.7, we know that the eight varieties in C besides c_M and c_M^* are precisely the varieties that share opposite corner triples (across the long diagonal) with c_M^* . We note that $|C'| = 20$, and that all cubes in C' share exactly two corner triples with c_M .

Given n varieties in the set C' , it is natural to ask how close those n varieties are to making a $2 \times 2 \times 2$ solution modeled after c_M . Specifically, each variety in C' can be used in two possible positions in the model. However, different varieties may share the same corner triple with c_M . As a result, it usually takes more than eight varieties from C' to make a solution modeled after c_M . We say that a subset S of varieties from C' is a *matching of size k* for the corner triples of c_M if k varieties from S can be used for k corner triples of a $2 \times 2 \times 2$ solution modeled after c_M . The following lemma determines the minimal size matching for different values of $V_{C'}$.

Lemma 4.4. *The table below gives the minimal size guaranteed matching on the corner triples of c_M for various values of $V_{C'}$.*

$V_{C'}$	1	2	3	4	5	6	7	8	≥ 9
Minimal matching	1	2	3	4	4	4	5	5	≥ 6

Proof. In the graph in Figure 4.1, the vertices represent the eight corner triples of c_M , as in Figure 3.1. For each variety c' in C' , we draw an edge between the two corner triples that c' shares with c_M . Recall from Lemma 2.7 that given two adjacent corner triples in c_M , there is exactly one other variety with those same corner triples. Similarly, for opposite corner pairs there are exactly two varieties that share those corner triples with c_M . Corner pairs appear in the graph below as bold edges. Alternatively, one can consider Figure 4.1 as an weighted graph, where edges corresponding to corner pairs have weight 2 whereas all other edges have weight 1. We note that the graph is more commonly known as the complete bipartite graph $K_{4,4}$.

To fill in the table in the lemma we determine, by observation, the minimal number of vertices incident to $V_{C'}$ arbitrary edges in the graph. We use this configuration to determine the minimal size of the matching. \square

Corollary 4.5. *If the size of the matching on the corner triples of c_M is greater than or equal to $8 - r(c_M)$, then one can construct a $2 \times 2 \times 2$ solution modeled after c_M .*

Proof. Use the matching to fill in $8 - r(c_M)$ appropriate corner triples on the solution. The $r(c_M)$ copies of c_M can be used in the remaining $r(c_M)$ positions. \square

We have a result with a similar flavor for the set C .

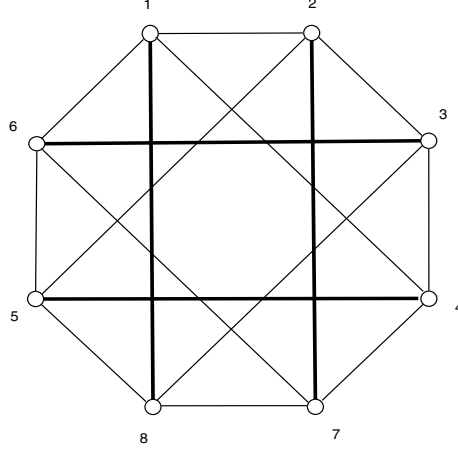


FIGURE 4.1. Graph of Shared Corner Triples

Lemma 4.6. *If $V_C \geq 9$, then there is a $2 \times 2 \times 2$ solution.*

Proof. The set C contains the varieties c_M , c_M^* , and the eight varieties that share no corner triples with c_M . By Lemma 2.7, each of these eight varieties shares two opposite corner triples with c_M^* . Since $V_C \geq 9$, there are at least eight other varieties in C besides c_M . If c_M^* is not one of these varieties, one each of the eight varieties can be used to construct a $2 \times 2 \times 2$ solution modeled after c_M^* . If c_M^* is one of the varieties, then the seven other varieties can be used as a matching for seven corner triples of a solution modeled after c_M^* . A cube of variety c_M^* completes the $2 \times 2 \times 2$ solution, also modeled after c_M^* . \square

For the rest of this section we fix the size of the set of cubes at 27, the size of the $3 \times 3 \times 3$ puzzle. The following lemma uses Lemma 4.1 and the number of varieties of cubes ($V_C + V_{C'}$) to determine solution criteria.

Lemma 4.7. *Given 27 cubes, if*

- $r(c_M) = 2$ and $V_C + V_{C'} \leq 20$
- $r(c_M) = 3$ and $V_C + V_{C'} \leq 15$
- $r(c_M) = 4$ and $V_C + V_{C'} \leq 11$

then there is a $2 \times 2 \times 2$ solution

Proof. Separate the 27 cubes by variety and let c_M be a variety with greatest repetition number. Let a_n be the number of distinct varieties that appear in the puzzle with multiplicity exactly n . Then $\sum a_n = V_C + V_{C'}$ and $\sum na_n = 27$. By subtracting the first equation from the second, we get the formula

$$(4.1) \quad \sum (n-1)a_n = 27 - (V_C + V_{C'}).$$

If $r(c_M) = k$ then we can rewrite Equation 4.1 as

$$(4.2) \quad (a_2 + \dots + a_k) + (a_3 + \dots + a_k) + \dots + (a_{k-1} + a_k) + (a_k) = 27 - (V_C + V_{C'}).$$

Notice that for $j > 1$, $a_j + a_{j+1} + \dots + a_k$ are the number of cube varieties that occur at least j times.

Assume that $r(c_M) = 2$. Then from Equation 4.2, $a_2 = 27 - (V_C + V_{C'})$. By Lemma 4.1, if $a_2 \geq 7$ then there is a $2 \times 2 \times 2$ solution, which is equivalent to $V_C + V_{C'} \leq 20$. Next, let $r(c_M) = 3$. By Equation 4.2, $(a_2 + a_3) + (a_3) = 27 - (V_C + V_{C'})$. By Lemma 4.1, if $(a_2 + a_3) + (a_3) \geq 12$ then there is a $2 \times 2 \times 2$ solution. Therefore,

$$12 \leq (a_2 + a_3) + (a_3) = 27 - (V_C + V_{C'})$$

or $V_C + V_{C'} \leq 15$. The case $r(c_M) = 4$ is identical. \square

Finally, we consider arbitrary sets of 27 cubes in cases according to the values of $r(c_M)$ and $V_C + V_{C'}$. The last four results give criteria for solutions based on the size of V_C , the size of $V_{C'}$, and the sizes of $V_C + V_{C'}$ and $r(c_M)$. These results handle most cases, although on occasion, we also need to invoke Lemma 4.1.

Theorem 4.8. *Given any set of 27 cubes, there is a subset of eight cubes that has a $2 \times 2 \times 2$ solution.*

Proof. If $r(c_M) = 1$, then $V_C + V_{C'} = 27$, which implies that either $V_C \geq 9$ or $V_{C'} \geq 19$. The result then follows either from Lemma 4.6 or from Figure 4.1 in Lemma 4.4.

Next assume that $r(c_M) \geq 2$. If $V_C + V_{C'} \geq 17$, then either $V_C \geq 9$ or $V_{C'} \geq 9$. If the former happens, there is a solution by Lemma 4.6. In the latter case, Corollary 4.5 implies the result. If $V_C + V_{C'} < 17$, then the result follows for $r(c_M) = 2$ by Lemma 4.7. Therefore, we may assume that $V_C + V_{C'} < 17$ and $r(c) \geq 3$

Cases $15 \leq V_C + V_{C'} \leq 16$: If $V_C \geq 9$ we are done by Lemma 4.6. Otherwise, $V_{C'} \geq 7$, and Lemma 4.4 implies that there is a matching for at least five of c_M 's corner triples. Since $r(c_M) \geq 3$, it follows that there is a solution by Corollary 4.5.

Cases $11 \leq V_C + V_{C'} \leq 14$: By Lemma 4.7 there is a solution when $r(c_M) \leq 3$, so we can assume that $r(c_M) \geq 4$. If $V_C \geq 9$ we are done by Lemma 4.6. Otherwise, $V_{C'} \geq 3$. If $V_{C'} \geq 4$ is also true, then Corollary 4.5 yields a solution. Otherwise, $V_{C'} = 3$, $V_C + V_{C'} = 11$, and $r(c_M) \geq 4$. The result follows from Lemma 4.7 (if $r(c_M) = 4$) and from Corollary 4.5 (if $r(c_M) > 4$).

Cases $V_C + V_{C'} \leq 10$: These cases cannot be handled as cleanly as the previous ones. Nonetheless, the result does hold by comparing the cases to the results in Lemma 4.1. Specifically, we may assume that $r(c_M) > 4$ by Lemma 4.7. There are some additional short cuts, although at this point it is also possible to simply enumerate all partitions of 27 elements into ten or fewer subsets and check for the existence of a solution case-by-case. Either way, one finds: when $r(c_M) = 5$, there must be six distinct triples, seven distinct doubles, or three distinct doubles and two distinct 4-tuples; when $r(c_M) = 6$ or $r(c_M) = 7$, then with four exceptions, there must be six distinct triples, seven distinct doubles, three distinct doubles and two distinct 4-tuples, or two distinct doubles and two distinct 6-tuples. In all these cases Lemma 4.1 guarantees a $2 \times 2 \times 2$ solution.

The four exceptions are the cases $(7, 7, 7, 1, 1, 1, 1, 1, 1, 1)$, $(7, 7, 6, 1, 1, 1, 1, 1, 1, 1)$, $(7, 5, 5, 4, 1, 1, 1, 1, 1, 1)$, and $(6, 6, 6, 5, 1, 1, 1, 1, 1, 1)$, where the vector entries are the number of cubes of each variety. These cases must be done ad hoc. Consider $(6, 6, 6, 5, 1, 1, 1, 1, 1, 1)$, with $r(c_M) = 6$. If any of the four varieties with repetition number greater than one share corner triples, then there is a $2 \times 2 \times 2$ solution. Assume this does not happen, and that the 32 corner triples on the four varieties are

all distinct. Of the remaining four varieties that each occur once (the singletons), if any two of them share corner triples with one of the cubes with repetition number six, or if any three of them share corner triples with the variety with repetition number five, then we are done by Corollary 4.5. We claim that this must happen. If not, then at most one of these singletons shares corner triples with the varieties with repetition number six and at most two singletons share corner triples with the variety with repetition number five. This implies that there are two singletons that share no more than two corner triples with the entire set of four varieties of high repetition number. As a result, these two singletons must have six corner triples from the set of eight remaining corner triples. Therefore, they have at least four corner triples in common, which implies that they must be the same variety, which is a contradiction. For each of the other three cases $r(c_M) = 7$. If no variety shares a corner triple with c_M , then $V_C \geq 9$ and we have a solution by Lemma 4.6. \square

This result implies our main theorem:

Theorem 4.9. *A $n \times n \times n$ puzzle always has a solution for $n \geq 3$.*

5. FURTHER QUESTIONS AND OBSERVATIONS

To get the results in our paper, we had to prove two main results: once a $2 \times 2 \times 2$ solution was found, there are always enough other cubes in the $n \times n \times n$ puzzle to complete the frame in some manner and hence complete the puzzle; and any set of 27 cubes suffices to find the corner solution. It is an interesting question as to when the frame can be completed using fewer than all n^3 cubes. This motivates the following definition:

Definition 5.1. *The frame number of the $n \times n \times n$ puzzle, $f(n)$, is the minimal number of cubes necessary to guarantee a solution to the puzzle's frame.*

In this language, our main result is

Theorem 5.2. *For $n > 2$, $f(n) \leq n^3$.*

How sharp is this result? The results in our paper show that $8 < f(2) \leq 27$ as the $2 \times 2 \times 2$ puzzle does not always have a solution, but every set of 27 cubes has a subset that forms a $2 \times 2 \times 2$ solution. In fact, we can do a little better.

Theorem 5.3. $22 < f(2) \leq 27$.

Proof. It is possible to find a set of five distinct varieties, no two of which contain any corner triples in common. Take seven cubes of any three of the varieties and one cube of another variety. There is no solution modeled after one of the four varieties because the varieties share no corner triples. On the other hand, there is no solution modeled after any other cube either, since each of the four varieties would have to contribute exactly two corner triples to the solution, and one of the varieties only has repetition number one. \square

In fact, we conjecture that $f(2) = 23$, although we have not been able to prove this. Theorem 5.3 shows why we had to work so hard to find a $2 \times 2 \times 2$ solution in a set of 27 cubes; the set is very close to the minimum possible size where a solution is guaranteed. We ask the following:

Open Questions:

- (1) What is the value of $f(n)$?
- (2) How much can the bound $f(n) \leq n^3$ be improved?
- (3) Is there a good lower bound for $f(n)$?

Weaker asymptotic versions of these questions can also be asked.

There is another related conjecture, motivated by the proof of Theorem 3.5 and the open questions. In Theorem 3.5 we showed that in an $n \times n \times n$ puzzle, a solution for the eight corner triples implies some solution for the frame. By looking carefully at the inequalities in the proof, one sees that many of them are fairly crude. Perhaps they can be tightened enough to prove the following:

Conjecture 5.4. *For n sufficiently large, $f(n) = 12n - 16$.*

As $12n - 16 = 12(n - 2) + 8$ is the number of cubes in the frame, a positive answer to this conjecture says that the (colored cubes)³ problem “should” be stated asymptotically as a statement about the frame of the cube. We note that a negative answer to the conjecture would also be interesting, as it would give some information about the complexity of the problem.

On a related note, in this paper we show that the puzzle has a solution for $n \geq 3$, and for $n \geq 6$ the solution is modeled after one of the puzzle cubes. However, our proof does not guarantee such a solution for arbitrary sets of cubes. For example, take seven cubes each from the set of five distinct varieties mentioned in Theorem 5.3, no two of which contain any corner triples in common. These 35 cubes contain a $2 \times 2 \times 2$ solution that can be complete to a solution of a $3 \times 3 \times 3$ puzzle, but clearly it is not modeled after any of the cubes in the set. How long does this happen? Specifically,

More Open Questions:

- (1) Does the $n \times n \times n$ puzzle have a solution modeled after a cube in the set of puzzle cubes for $n = 4, 5$?
- (2) For an $n \times n \times n$ puzzle with $n = 3, 4, 5$, how large a set of cubes is required so that a solution to the frame is modeled after a cube in the set?

Our comments so far have dealt solely with the (colored cubes)³ problem as stated. There are other natural generalizations of our puzzle. One simply uses more colors. One can keep the regularity condition (distinct faces have distinct colors), or consider the problem with greatest generality where any coloring of a cube is allowed. An implementation similar to the one mentioned after Lemma 4.1, using more cubes and corner triples, might be helpful in these cases. Some readers might have noticed a similarity between our implementation and a block design. In fact, we initially considered analyzing the (colored cubes)³ problem by considering it as a block design. However, it is possible for two corner triples to never occur on the same cube (the corner triples $(1, 2, a)$ and $(1, 2, b)$ with $a \neq b$, for example) so the design is not balanced. Perhaps a more general version of the problem could be described in terms of some balanced block design.

For other generalizations, one could also attempt a (concentric colored cubes)³ problem, where all of the concentric cubes must be solved in order for the puzzle to be solved. Or, one could try the mind-bending version of the puzzle in dimension d for $d > 3$. One would start with n^d d dimensional hypercubes and a palette of $2d$ colors. The solved puzzle would have $2d$ monochromatic faces.

We end with a remark about the running time of our puzzle algorithm. The solution to our problem as stated is $O(n)$ in the number of puzzle cubes. Briefly,

one picks a subset of 27 cubes and finds a $2 \times 2 \times 2$ solution ($O(1)$). Next, the cubes have to be sorted into distinct varieties ($O(n)$). Theorem 3.5 describes how to determine the cubes to be used for the frame ($O(n)$ at worst). On the other hand, in light of Robertson and Munro's result, we would not be at all surprised if the solution to the most general (colored cubes)³ problem turned out to be, like Instant Insanity, NP-hard. Clearly, there are a number of interesting questions related to colored cube puzzles still to be investigated.

Acknowledgments: We would like to thank Garth Isaak for a conversation that motivated this research problem. We would also like to thank Gary Gordon and the referees, whose suggestions greatly improved the presentation of this paper.

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