THE COHOMOLOGY GROUPS OF THE OUTER WHITEHEAD AUTOMORPHISM GROUP OF A FREE PRODUCT

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1. Introduction

Fix a free product decomposition $\Gamma = G_1 * G_2 * \cdots * G_n$, where the G_i are freely indecomposable groups, and fix an index i along with a non-identity element $g_j \in G_j$ (allowing the possibility that $i = j$). Then the automorphism $\alpha_i^{g_j} \in \mathrm{Aut}(\Gamma)$ induced by

$$\alpha_i^{g_j}(g_k) = \begin{cases} g_k & \text{if } k \neq i \\ g_k^{g_j} & \text{if } k = i \end{cases}$$

(where $g_k \in G_k$ and $g_k^{g_j}$ is shorthand for conjugating g_k by g_j) is an *elementary Whitehead automorphism*. The *Whitehead automorphism group*, $\mathrm{Wh}(\Gamma)$, is the subgroup of $\mathrm{Aut}(\Gamma)$ generated by the $\alpha_i^{g_j}$. If none of the G_i are infinite cyclic, then $\mathrm{Wh}(\Gamma)$ is the kernel of the map $\mathrm{Aut}(\Gamma) \twoheadrightarrow \mathrm{Out}(G_1 \times \cdots \times G_n)$. In particular, if the G_i are finite, then $\mathrm{Wh}(\Gamma)$ is a finite-index subgroup of $\mathrm{Aut}(\Gamma)$.

In this paper we compute the cohomology groups of $\mathrm{OWh}(\Gamma)$, the quotient of $\mathrm{Wh}(\Gamma)$ in the outer-automorphism group, with field coefficients. Our approach is to analyze the equivariant spectral sequence associated to the action of $\mathrm{OWh}(\Gamma)$ on a contractible, simplicial complex introduced by McCullough and Miller [10]. Throughout the paper, cohomology groups will be assumed to have field coefficients unless otherwise indicated, and $(\Gamma)^n$ denotes the n-fold product $\underbrace{\Gamma \times \cdots \times \Gamma}_{n \text{ copies}}$.

Main Theorem. *Let $\Gamma = G_1 * G_2 * \cdots * G_n$ where each G_i is of type FP_∞. Then the i^{th} cohomology group of $\mathrm{OWh}(\Gamma)$, with field coefficients, is*

$$H^i(\mathrm{OWh}(\Gamma)) \simeq H^i((\Gamma)^{n-2}).$$

The Main Theorem states an additive isomorphism between the cohomology groups of $\mathrm{OWh}(\Gamma)$ with the cohomology groups of a direct product of an appropriate number of copies of Γ. We extend this result to the case where the factor groups are abelian in §5 and we then compute the ring structure of $\mathrm{Wh}(W_n)$, where

$$W_n = \underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{n \text{ copies}}$$

is a free Coxeter group, in §6. The cohomology algebra of $\mathrm{Wh}(W_n)$, over the field with two elements, is generated by $n(n-1)$ one dimensional classes, α_{ij}^*, $1 \leq i, j \leq n$ and $i \neq j$, subject to the relations:

1. $\alpha_{ij}^* \alpha_{ji}^* = 0$;

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2. $(\alpha^*_{kj})^n(\alpha^*_{ji})^m = \Big((\alpha^*_{kj})^n - (\alpha^*_{ij})^n\Big)(\alpha^*_{ki})^m$ for all m, n.

As a consequence we establish, in Theorem 6.3, that while there is an additive isomorphism, there is no ring-isomorphism between the cohomology of $\mathrm{W\scriptstyle{H}}(W_n)$ and the cohomology of $(W_n)^{n-1}$.

Our argument builds directly from previous arguments, namely the computation of the Euler characteristic of $\mathrm{W\scriptstyle{H}}(\Gamma)$ given in [7], a cohomology ring calculation in [3], and the analysis of a spectral sequence given in [6]. In fact, our main results are analogs of the results in [6]. The reader unfamiliar with these papers is strongly encouraged to read them first. We have restricted ourselves to field coefficients (usually suppressed in the notation) as a matter of convenience, since our computations involve the cohomology groups of direct products. We know of no evidence that suggests the isomorphisms do not hold more generally.

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2. The McCullough-Miller Complex

Underlying this work is the action of $\mathrm{OW\scriptstyle{H}}(\Gamma)$ on a contractible complex constructed by McCullough and Miller in [10]. In this section we remind the reader of this action; the reader interested in the details of this action may wish to consult earlier papers, such as [8], where this material is developed in greater detail.

Given $\Gamma = G_1 * G_2 * \cdots * G_n$ there is an associated contractible simplicial complex MM on which $\mathrm{OW\scriptstyle{H}}(\Gamma)$ acts cocompactly. The complex MM is the geometric realization of a poset whose elements consist of certain actions of Γ on trees. Any such action can be described by its fundamental domain, hence vertices in MM can be thought of as being associated to certain finite trees, labeled by the various isotropy groups, modulo an equivalence relation coming from the choices involved in picking a fundamental domain for an action.

Our computation only uses the fundamental domain for the action of $\mathrm{OW\scriptstyle{H}}(\Gamma)$ on MM. This fundamental domain has been described as the geometric realization of a certain poset of trees (referred to as the *Whitehead poset* in [10]) and as the geometric realization of a poset of hypertrees (see [8]). Here we use the McCullough-Miller description of this fundamental domain.

Definition 2.1. An $[n]$-*tree* is a bipartite tree where:

1. One set of vertices is labeled by $[n] = \{1, \ldots, n\}$;
2. The other set of vertices is unlabeled; and
3. All leaves are labeled.

We say that $\mathcal{T} < \mathcal{T}'$ if $\mathcal{T}$ can be created from $\mathcal{T}'$ by folding $\mathcal{T}'$ at labeled vertices. For example, the leftmost tree in Figure 1 is formed from the center one by folding over the vertex labeled 4 and identifying the two edges incident to 4 and their adjacent unlabeled vertices. In a similar way, the center tree comes from the rightmost one by folding together the two edges to the right of the vertex labeled 4.

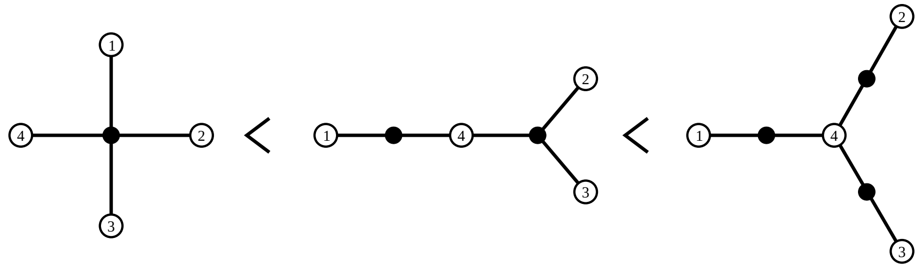

Figure 1. Three examples of [4]-trees, with the indicated partial order.

Following [8] we denote the poset of $[n]$-trees partially ordered by folding by HT_n. Figure 1 shows a maximal chain of three [4]-trees in HT_4. This chain corresponds to a 2-simplex in the geometric realization.

Proposition 2.2. *The fundamental domain for* $\mathrm{OW}\textsc{h}(\Gamma) \curvearrowright \mathrm{MM}$ *is the geometric realization of* HT_n*. It is a strong fundamental domain, meaning the fundamental domain is isomorphic to the quotient* $\mathrm{OW}\textsc{h}(\Gamma)\backslash\mathrm{MM}$*.*

The stabilizers of simplices in the fundamental domain HT_n are easy to describe.

Proposition 2.3 (Proposition 5.1 in [10])**.** *Let $\mathcal{T}$ be an $[n]$-tree. Denote the degree of the vertex labeled i in $\mathcal{T}$ by δ_i. Then the stabilizer of $\mathcal{T} \in \mathrm{HT}_n$ under the action of* $\mathrm{OW}\textsc{h}(\Gamma)$ *is*

$$\textsc{Stab}(\mathcal{T}) = G_1^{\delta_1 - 1} \times \cdots \times G_n^{\delta_n - 1} .$$

Further, if $\mathcal{T}_0 < \cdots < \mathcal{T}_k$ is a chain in HT_n, then the stabilizer of the associated k-simplex in MM *is* $\textsc{Stab}(\mathcal{T}_0)$*.*

For example, the stabilizers corresponding to the elements of MM indicated in Figure 1 are: trivial, G_4, and $G_4 \times G_4$, respectively.

Remark 1. The *rank* of an $[n]$-tree $\mathcal{T}$ is defined to be one less than the number of unlabeled vertices of the tree. (Going left to right in Figure 1, the [4]-trees have rank 0, 1, and 2.) It is not hard to show that the number of factors in the description of the stabilizer of $\mathcal{T}$, $r = \sum_{i=1}^{n}(\delta_i - 1)$, is the rank of the hypertree. This follows from Euler's formula for a tree, (# of vertices) − (# of edges) = 1, using the $[n]$-tree's description as a bipartite graph.

Example 2.4. The poset HT_3 is particularly simple. There are only four distinct [3]-trees and the geometric realization of HT_3 is a finite tree (shown in Figure 2). Using Proposition 2.3 we see that the stabilizer of the central vertex under the action of $\mathrm{OW}\textsc{h}(G_1 * G_2 * G_3)$ is trivial; the stabilizers of the other three vertices are G_1, G_2 and G_3. Since the associated complex MM is contractible, it must be a tree composed of copies of the tripod $|\mathrm{HT}_3|$. In MM, all vertices corresponding to rank 0 trees are trivalent, while the vertices corresponding to rank 1 trees have degree equal to the order of the group G_i, depending on the labeling. As was noted in [10], this shows that $\mathrm{OW}\textsc{h}(G_1 * G_2 * G_3) \simeq G_1 * G_2 * G_3$. It follows that $\mathrm{W}\textsc{h}(G_1 * G_2 * G_3)$ is a '$(G_1 * G_2 * G_3)$–by–$(G_1 * G_2 * G_3)$' group.

The situation for $n > 3$ is considerably more complicated. A picture of the fundamental domain for $\mathrm{OW}\textsc{h}(\Gamma) \curvearrowright \mathrm{MM}$ when $n = 4$ (HT_4) can be found in [6] and [10].

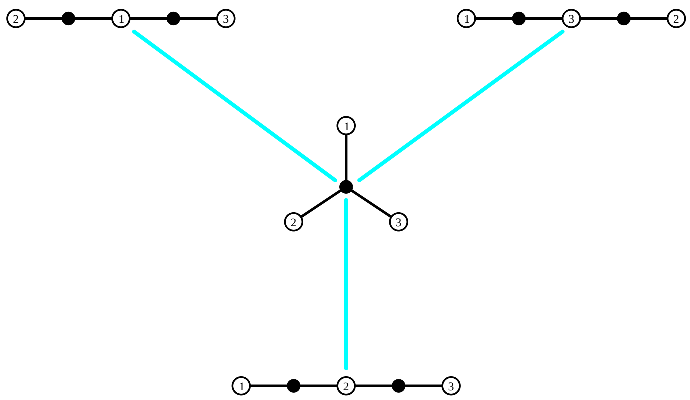

Figure 2. The geometric realization of HT_3 is the fundamental domain for the action of $\mathrm{OW}\textsc{h}(G_1 * G_2 * G_3)$ on MM is a tripod.

In previous papers ([6], [7], and [8] for example) switching from $[n]$-trees, as described above, to hypertrees made certain combinatorial arguments easier. As we only use the conclusions of these arguments, we do not make this switch in terminology in this paper.

3. The Equivariant Spectral Sequence

Given an action of a group G on a contractible simplicial complex Δ, there is an associated spectral sequence, usually called the equivariant spectral sequence. The E_1 page of this spectral sequence is given by

$$E_1^{pq} = \prod_{\sigma \in \Delta_p} H^q(\textsc{Stab}(\sigma), \mathfrak{K}) \Rightarrow H^{p+q}(G, \mathfrak{K})$$

where Δ_p denotes the set of p-simplices in a chosen fundamental domain for $G \curvearrowright \Delta$. The d_1 differential in this spectral sequence is a combination of coboundary maps and restriction maps to subgroups, which in this case are direct summands as the stabilizer groups are direct products. (See §VII.7 and §VII.8 of [2].) In our setting, p-simplices in the fundamental domain $|\mathrm{HT}_n|$ are associated to $(p+1)$-chains of $[n]$-trees. The stabilizer of a simplex in $|\mathrm{HT}_n|$ is the stabilizer of the smallest element in the chain (Proposition 2.3). Hence the first page of this spectral sequence is relatively easy to construct for very small values of n. An example is shown in Figure 3.

The equivariant spectral sequence for $\mathrm{OW}\textsc{h}(\mathbb{F}_n) \curvearrowright \mathrm{MM}$, where $\mathbb{F}_n$ is the free group on n generators, was analyzed in [6] to determine the integral cohomology $H^*(\mathrm{OW}\textsc{h}(\mathbb{F}_n), \mathbb{Z})$. Since our argument builds in a significant way on that calculation, we summarize it in the next few paragraphs.

When $\Gamma = \mathbb{F}_n$, then $G_i \simeq \mathbb{Z}$ for all i and simplex stabilizers are free abelian groups of rank less than or equal to $n-2$ by Proposition 2.3 and Remark 1. From the facts that the complex $|\mathrm{HT}_n|$ has geometric dimension $n-2$ and $\mathbb{Z}^{n-2}$ has cohomological dimension $n-2$, it follows that the E_1 page of the spectral sequence for $H^*(\mathrm{OW}\textsc{h}(\mathbb{F}_n), \mathbb{Z})$ is trivial outside of $[0, n-2] \times [0, n-2]$. In particular, it follows immediately that $H^*(\mathrm{OW}\textsc{h}(\mathbb{F}_n), \mathbb{Z})$ has finite cohomological dimension.

$$
\begin{array}{lll}
\prod_i H^3(G_i^2) \times \prod_{i,j} H^3(G_i \times G_j)^2 & \prod_i H^3(G_i)^9 & 0 \\
\prod_i H^2(G_i^2) \times \prod_{i,j} H^2(G_i \times G_j)^2 & \prod_i H^2(G_i)^9 & 0 \\
\prod_i H^1(G_i^2) \times \prod_{i,j} H^1(G_i \times G_j)^2 & \prod_i H^1(G_i)^9 & 0 \\
\mathfrak{K}^{29} & \mathfrak{K}^{64} & \mathfrak{K}^{36}
\end{array}
$$

FIGURE 3. The E_1 page of the equivariant spectral sequence for the action $\mathrm{OW}\textsc{h}(G_1 * \cdots * G_4) \curvearrowright \mathrm{MM}_4$. Except in the bottom row, the field coefficients have been suppressed. See [6] and [10] for an explicit description of the stabilizers.

The number of simplices in $|\mathrm{HT}_n|$ grows super-exponentially as n increases, a priori making direct calculation of the spectral sequence extremely difficult. It is a significant result in [6] that one can completely analyze the E_1 page regardless. A careful analysis of HT_n and its interaction with the d_1 differential shows that the E_1 page splits into a union of two types of chain complexes. One type corresponds to the augmented chain complex of a contractible subcomplex of $|\mathrm{HT}_n|$. This chain complex has trivial cohomology, and hence does not contribute any classes to the E_2 page of the spectral sequence. The second type corresponds to the chain complex of a contractible subcomplex of $|\mathrm{HT}_n|$ without augmentation. The cohomology of this type of complex consists of a single class in dimension 0. Hence, these complexes contribute a single class to the zero column of the E_2 page of spectral sequence.

Therefore, all cohomology classes on the E_1 page which survive to the E_2 page are in the zero column. This implies that $E_2 = E_\infty$ and the spectral sequence collapses. Furthermore, there is a bijective correspondence between the chain complexes which contribute classes to the E_2 page and certain $[n]$-trees. These trees are called *essential* in [6], and each rank q essential tree contributes a $\mathbb{Z}$-summand in $H^q(\mathrm{OW}\textsc{h}(\mathbb{F}_n), \mathbb{Z})$. Once the essential trees are enumerated by rank, the additive cohomology calculation in [6] follows.

The cases we are considering are complicated by the fact that G_i may not be isomorphic to G_j (hence more care needs to be taken in tracking the individual G_i's). Further, the cohomology groups on the first page of the spectral sequence may be more intricate than the entries that appeared in the free group case. For example, $H^*(G_i)$ may have non-trivial classes in arbitrarily high dimension, as in the case where each G_i is finite. Hence, while the equivariant spectral sequences we are considering are still first quadrant spectral sequences, they may not be bounded. Nonetheless, the cohomology classes that do survive to E_∞ are closely related to the ones associated to the essential $[n]$-trees in the calculation of $H^*(\mathrm{OW}\textsc{h}(\mathbb{F}_n), \mathbb{Z})$.

The entries in the equivariant spectral sequence are products of cohomology groups $H^k(G_1^{\delta_1 - 1} \times \cdots \times G_n^{\delta_n - 1})$, paired with a simplex σ. The group $G_1^{\delta_1-1} \times \cdots \times G_n^{\delta_n-1}$ is the stabilizer of σ, and also of the minimal tree $\mathcal{T}$ in the chain associated to σ. As we are using field coefficients, the Künneth formula says every element of $H^k(G_1^{\delta_1-1} \times \cdots \times G_n^{\delta_n-1})$ can be viewed as a sum of products of cohomology classes from the various cohomology groups $H^*(G_i)$. That is, every term in a given entry of the equivariant spectral sequence can be expressed as a sum of monomials: $x = \sum(\times_{i=1}^r x_i)$, where each x_i belongs to $H^j(G_k)$ for some k and r is the rank of $\mathcal{T}$.

Let $x = \times_{i=1}^r x_i$ be a monomial class in the E_1 page with an associated tree $\mathcal{T}$ and simplex σ in the geometric realization of HT_n. Then this entry has an analogous entry in the spectral sequence associated to $H^*(\mathrm{OW{\scriptsize H}}(\mathbb{F}_n);\mathbb{Z})$, where one focuses on the same tree $\mathcal{T}$ and simplex σ, and a monomial class $y = \times_{i=1}^r y_i$, where each y_i is a generator of $H^1(\mathbb{Z};\mathbb{Z})$ if $x_i \neq 1$, and is the identity otherwise. It is noted in [6] that this entry is part of an integral chain complex contained in the E_1 page of the spectral sequence, where the d_1 differential defines the boundary map. This chain complex is the either the augmented or unaugmented chain complex associated to a contractible subcomplex of $|\mathrm{HT}_n|$. Thus it either contributes a $\mathbb{Z}$ to the zero column of the E_2 page, or it contributes no classes at all to the E_2 page. Further, all chain complexes which contribute a $\mathbb{Z}$ to the zero column have a corresponding class $y = \times_{i=1}^r y_i$ where each $y_i \neq 1$. We note one main difference between the chain complex in the spectral sequence in the case of $\Gamma = G_1 * \cdots * G_n$ and $\mathbb{F}_n = \mathbb{Z} * \cdots * \mathbb{Z}$: since we have no a priori control over the dimension of the x_i's, the entry corresponding to $x = \times_{i=1}^r x_i$ may occur in a higher row than the entry corresponding to its analog $y = \times_{i=1}^r y_i$.

The entry we are considering in the spectral sequence, $x = \times_{i=1}^r x_i$, similarly determines a chain complex associated to the same subcomplex of $|\mathrm{HT}_n|$, but with coefficients in the field $\mathfrak{K}$. That is, the chain complex containing x is simply the tensor product of the integral chain complex for y with $\mathfrak{K}$. Thus this complex either contributes a single $\mathfrak{K}$ to the zero column of the E_2 page, or it does not survive. We may then use the characterization of which chain complexes contribute to the E_2 page given in [6]:

Definition 3.1. A monomial cohomology class x of $H^*(\mathrm{S{\scriptsize TAB}}(\mathcal{T}), \mathfrak{K})$ in the zero column of the spectral sequence is an *essential class* if

1. $\mathcal{T}$ is an essential $[n]$-tree;
2. the restriction in cohomology of x to each summand of $\mathrm{S{\scriptsize TAB}}(\mathcal{T})$ is non-trivial.

(The definition of an 'essential' $[n]$-tree is recalled in the next section.)

Theorem 3.2. *The spectral sequence for the calculation of $H^*(\mathrm{OW{\scriptsize H}}(\Gamma))$ collapses at the E_2 page and consists entirely of classes in the 0 column corresponding to essential classes.*

4. Counting Planted Forests

From the analysis of the spectral sequence in the last section, the determination of the additive structure of $H^*(\mathrm{OW{\scriptsize H}}(\Gamma))$ has been reduced to a combinatorial argument counting essential classes. These classes are associated to essential n-trees, whose definition we recall from [6].

Definition 4.1. An $[n]$-tree is *essential* if there is an unlabeled vertex x, adjacent to the vertex labeled 1, such that:

1. No unlabeled vertex other than x has degree > 2;
2. If x is removed from the tree then the vertices labeled 1 and 2 are in separate connected components.

For consistency with [6] we refer to the vertex x as the *thick* vertex.

Our counting can be facilitated by identifying essential $[n]$-trees with certain planted forests on $[n]$. Namely, in Figure 4 we indicate how essential $[n]$-trees correspond to *essential planted forests* on $[n]$, where in order for a forest to be 'essential' one needs:

1. The vertex labeled 1 to be a root;
2. The vertex labeled 2 to be in a tree that is not rooted at the vertex labeled 1.

The correspondence is given, in one direction, by rooting an essential tree at its thick vertex x, removing this root, the edges attached to the root, and then replacing all barycentrically subdivided edges by undivided edges.

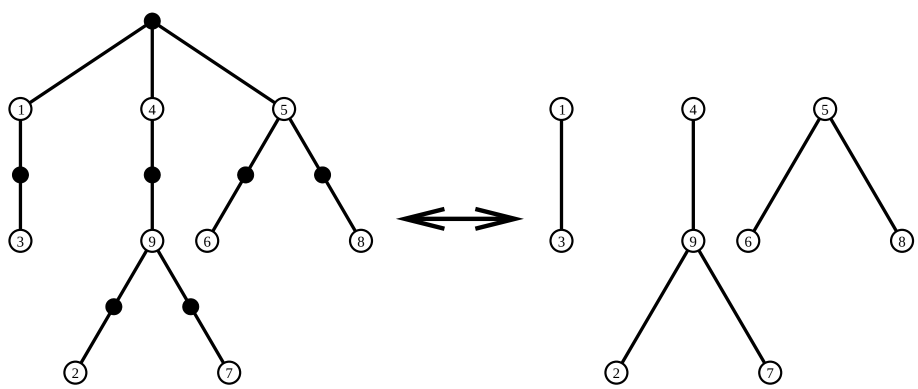

FIGURE 4. Essential $[n]$-trees correspond to essential planted forests.

The number of essential forests is known and is given in [6]. However, that answer is insufficient for our case. For $\mathrm{OW}\textsc{h}(\mathbb{F}_n)$ the essential classes were in one-to-one correspondence with the essential forests. In our situation the complicating factor is that each G_i may have cohomology classes in arbitrarily high dimension. Thus, given a fixed $[n]$-tree $\mathcal{T}$ and its stabilizer $\textsc{Stab}(\mathcal{T}) = G_1^{\delta_1 - 1} \times \cdots \times G_n^{\delta_n - 1}$, a cohomology class of the form $x = \times_{i=1}^{r} x_i$, can potentially appear in any dimension. (This could not happen in the case considered in [6], since in that paper each $G_i \cong \mathbb{Z}$.) Therefore, an essential $[n]$-tree may have an infinite number of essential classes associated to it. To surmount this obstacle we stratify the count and only consider essential forests with a particular degree sequence on the vertices. This gives us enough differentiation to keep track of cohomology classes in each dimension on the E_∞ page of the spectral sequence discussed in the previous section.

Define the *descending degree* of a vertex v in an essential forest to be the number of edges incident to the vertex in the direction away from the root. Therefore, the descending degree of a vertex is one less than its degree when a vertex is not the root and the same as its degree when it is the root. (The descending degree of the vertices labeled 5 and 9, in the forest in Figure 4, is 2.) Since the vertices are labeled by $[n]$, there is a finite *sequence of descending degrees*, $\{d_1, \ldots, d_n\}$, where d_i is the descending degree of the vertex labeled i.

Given an $[n]$-tree $\mathcal{T}$, there is a relationship between δ_i, the degree of the vertex labeled i in the tree and d_i, the descending degree of the vertex labeled i in the associated planted forest. Specifically, since there are edges between the unlabeled thick vertex and the roots of the associated planted forest, $d_i = \delta_i - 1$. Therefore, the first part of Proposition 2.3 can be restated as saying that if the descending degree of the vertex labeled i is d_i, then the stabilizer of the associated essential $[n]$-tree is $G_1^{d_1} \times \cdots \times G_n^{d_n}$.

The following sequence of lemmas shows that the number of essential planted forests with descending degree sequence $\{d_1, \ldots, d_n\}$ is given by the multinomial coefficient $\binom{n-2}{d_1, \ldots, d_n}$. We note that if $d_1 + \ldots + d_n + l = n$, then the multinomial coefficient is given by

$$\binom{n}{d_1, \ldots, d_n} = \binom{n}{d_1, \ldots, d_n, l} = \frac{n!}{d_1! \cdots d_n!\ l!}.$$

We use the following result a number of times in the rest of the paper.

Theorem 4.2. *(Theorem 5.3.4 in* [11]*). Let $\{d_1, \ldots, d_m\}$ be a descending degree sequence with $\sum d_i = m - l$. Then the number of planted forests on $[m]$ is*

$$\binom{m-1}{l-1}\binom{m-l}{d_1, d_2, \ldots, d_m}.$$

Lemma 4.3. *There is a $\binom{d_1+k-1}{d_1}$-to-1 map from planted forests with d_1+k-1 components and vertices labeled by $\{2, \ldots, n\}$ to k component planted forests on $[n]$ where 1 is a root.*

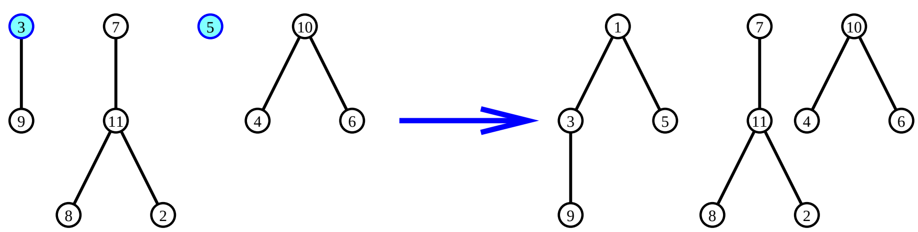

FIGURE 5. Counting planted forests

Proof. Take a planted forest on $\{2, 3, \ldots, n\}$ with more than d_1 components. By choosing any d_1 components to be subtrees for the vertex labeled 1, we get a planted forest on $[n]$ with $d_1 - 1$ fewer components. (See Figure 5.) □

Lemma 4.4. *The number of planted forests on $[n]$ with 1 as a root and given degree sequence $\{d_1, \ldots, d_n\}$ is*

$$\binom{d_1+k-1}{d_1}\binom{n-2}{d_1+k-2}\binom{n-k-d_1}{d_2, \ldots, d_n}$$

where $k = n - \sum d_i$ is the number of components.

Proof. We first note that the number of components in a planted forest on $[n]$ with descending degree sequence $\{d_1, \ldots, d_n\}$ is the Euler characteristic, hence it is $k = n - \sum d_i$.

Removing the vertex labeled 1 from a forest satisfying the hypotheses of the lemma results in a forest on $\{2, 3, \ldots, n\}$ with $d_1 + k - 1$ components. Apply Theorem 4.2 with $m = n - 1$ and $l = d_1 + k - 1$ to count all forests on $\{2, 3, \ldots, n\}$ with descending degree sequence $\{d_2, \ldots, d_n\}$. Then multiply by the multiplicity of the map given in Lemma 4.3, and the result follows. □

Lemma 4.5. *The number of essential forests on $[n]$ with given descending degree sequence d is $\binom{n-2}{d_1, \ldots, d_n}$.*

Proof. Lemma 4.4 says there is a one-to-one correspondence between "planted forests on $[n]$ with 1 as a root and with descending degree sequence $d = \{d_1, \ldots, d_n\}$" and pairs consisting of a planted forest $\mathcal{F}$ on $\{2, \ldots, n\}$ with degree sequence $\{d_2, \ldots, d_n\}$ together with a choice of d_1 components of $\mathcal{F}$. Of all the possible $\binom{d_1+k-1}{d_1}$ ways of grafting the vertex labeled 1 onto $\mathcal{F}$, the fraction

$$\frac{\binom{d_1+k-2}{d_1}}{\binom{d_1+k-1}{d_1}} = \frac{k-1}{d_1+k-1}$$

yields a forest where the vertex labeled by 2 is not in the tree rooted at 1. Thus by Lemma 4.4 the number of essential forests on $[n]$ with given descending degree sequence d is given by

$$\begin{aligned} &\frac{k-1}{d_1+k-1} \cdot \binom{d_1+k-1}{d_1}\binom{n-2}{d_1+k-2}\binom{n-k-d_1}{d_2, \ldots, d_n} \\ &\quad = \frac{k-1}{d_1+k-1} \cdot \frac{(d_1+k-1)!}{d_1!(k-1)!} \cdot \frac{(n-2)!}{(d_1+k-2)! \cdot (n-k-d_1)!} \cdot \frac{(n-k-d_1)!}{d_2! \cdots d_n!} \\ &\quad\quad = \frac{(n-2)!}{d_1! d_2! \cdots d_n! (k-2)!} = \binom{n-2}{d_1, \ldots, d_n, k-2} = \binom{n-2}{d_1, \ldots, d_n}. \end{aligned}$$

□

The proof of our main result is now straightforward.

Main Theorem. *Let $\Gamma = G_1 * G_2 * \cdots * G_n$ where each G_i is of type FP_∞. Then the i^{th} cohomology group of $\mathrm{OW{\small H}}(\Gamma)$, with field coefficients, is*

$$H^i(\mathrm{OW{\small H}}(\Gamma)) \simeq H^i((\Gamma)^{n-2}).$$

Proof. To exhibit the isomorphism we count the number of classes in dimension i of a particular type from the cohomology of both groups and show that the counts are equal. We start with $H^i(\mathrm{OW{\small H}}(\Gamma))$. Let x be an essential class in dimension i. From Definition 3.1, the class x is associated to an essential $[n]$-tree and can be written as $x = \times_{i=1}^{r} x_i$ where each x_i is a non-identity monomial class in some $H^*(G_j)$. Say that d_k of the classes x_i come from $H^*(G_k)$, and assume for the time being that $x_i \neq x_j$ when $i \neq j$. In the spectral sequence, the class x will be paired with each essential forest with descending degree sequence $\{d_1, \ldots, d_n\}$.

Furthermore, for each such essential forest there are exactly $d_1!d_2!\dots d_n!$ classes isomorphic to x, as the d_k different classes from $H^*(G_k)$ can be chosen in $d_k!$ ways. Therefore, by Lemma 4.5 there are

$$\binom{n-2}{d_1,\dots,d_n} d_1!d_2!\dots d_n!$$

essential classes in the spectral sequence isomorphic to x.

On the other hand, this is the same as the number of times a class isomorphic to x will appear in $H^i((\Gamma)^{n-2})$. Starting with the product $(\Gamma)^{n-2}$, pick d_1 copies of Γ from which to take terms from $H^*(G_1)$, d_2 of the remaining copies of Γ from which to take terms from $H^*(G_2)$, etc. The total number of ways to pick appropriate factors from $(\Gamma)^{n-2}$ to generate a class isomorphic to x is the multinomial coefficient $\binom{n-2}{d_1,\dots,d_n}$. Once the appropriate factors are chosen, the d_j classes from $H^*(G_j)$ can be chosen in $d_j!$ ways, completing the equivalence.

We note that if $x_i = x_j$ for some $i \neq j$, then the counting argument goes through in the same way as above, although the notation gets significantly more complicated. $\square$

5. A Partial Extension to Wh(Γ)

Once the cohomology groups of $\mathrm{OW{\scriptstyle H}}(\Gamma)$ are known, it is possible to determine the cohomology groups of $\mathrm{W{\scriptstyle H}}(\Gamma)$, since the short exact sequence

$$1 \to \Gamma \to \mathrm{W{\scriptstyle H}}(\Gamma) \to \mathrm{OW{\scriptstyle H}}(\Gamma) \to 1$$

has an associated fiber bundle $p : B\mathrm{W{\scriptstyle H}}(\Gamma) \to B\mathrm{OW{\scriptstyle H}}(\Gamma)$ with fiber $B\Gamma$ which gives a fibration (see Theorems 1.6.11 and 2.4.12 of [1]).

We remind the reader of the Leray-Hirsch Theorem, as presented in [5]:

Theorem 5.1 (Leray-Hirsch Theorem)**.** *Let $F \xrightarrow{\iota} E \xrightarrow{\rho} B$ be a fiber bundle such that*

1. *$H^n(F, \mathfrak{K})$ is a finitely generated free $\mathfrak{K}$-module for each n, and*
2. *There exist classes $c_j \in H^{k_j}(E, \mathfrak{K})$ whose restrictions $\iota^*(c_j)$ form a basis for $H^*(F, \mathfrak{K})$ in each fiber F.*

Then the map $\Phi : H^*(B, \mathfrak{K}) \otimes H^*(F, \mathfrak{K}) \to H^*(E, \mathfrak{K})$ given by

$$\sum_{i,j} b_i \otimes \iota^*(c_j) \mapsto \sum_{i,j} \rho^*(b_i) \cup c_j$$

is an isomorphism.

Remark 2. One can show that the second hypothesis of the Leray-Hirsch Theorem implies that in the Serre spectral sequence associated to the fiber bundle, $F \xrightarrow{\iota} E \xrightarrow{\rho} B$, the system of local coefficients on B is trivial. This condition, which usually appears in statements of the Leray-Hirsch Theorem, implies that the spectral sequence collapses at the E_2 page. [9]

The first hypothesis in Theorem in 5.1 comes for free since we are using field coefficients and we are assuming that each G_i is of type FP_∞.

For the second hypothesis, the kernel of the short exact sequence above is a free product, $\Gamma = G_1 * G_2 * \cdots * G_n$, so to get a basis for $H^*(\Gamma)$ it suffices to get a basis for each $H^*(G_i)$. Notice that if each G_i is abelian, then there is a homomorphism

$\pi_i : \mathrm{W\small{H}}(\Gamma) \to G_i$ induced by $\phi(\alpha_j^{g_i}) = g_i$, where j is a fixed index different from i, and $\phi(\alpha_k^{g_l}) = 1$ whenever $k \neq j$ or $i \neq l$. This can be verified by consulting a presentation for $\mathrm{W\small{H}}(\Gamma)$ (see [4]). We must assume the G_i are abelian because $\alpha_i^{h_i}\alpha_j^{g_i} = \alpha_j^{{g_i}^{h_i}}\alpha_i^{h_i}$, for $g_i, h_i \in G_i$. (This is Relation V in [4].) Thus, if the map π_i exists, then it must be the case that $g^h = g$ for all $g, h \in G_i$, that is, that G_i is abelian.

Let $\iota_i : G_i \to \mathrm{W\small{H}}(\Gamma)$ denote the map defined by the two injections

$$G_i \hookrightarrow \Gamma = G_1 * G_2 * \cdots * G_n \hookrightarrow \mathrm{W\small{H}}(\Gamma)$$

where the second map sends g_i to $\alpha_1^{g_i}\alpha_2^{g_i}\cdots\widehat{\alpha_i^{g_i}}\cdots\alpha_n^{g_i}$. The composition $\pi_i \circ \iota_i$ is the identity on G_i which implies that the restriction map on cohomology, ι_i^*, is onto.

Having satisfied the hypotheses, in the case where the G_i are abelian, we may apply the Leray-Hirsch Theorem to obtain:

Lemma 5.2. *Let Γ be a free product of abelian groups and let $\{c_s\}$ be a collection of classes in $H^*(\mathrm{W\small{H}}(\Gamma))$ that map via the restriction map isomorphically onto a basis for $H^*(\Gamma)$. Then the map*

$$H^*(\mathrm{OW\small{H}}(\Gamma)) \otimes H^*(\Gamma) \to H^*(\mathrm{W\small{H}}(\Gamma))$$

defined by

$$\sum_{j,s} b_j \otimes i^*(c_s) \mapsto \sum_{j,s} p^*(b_j) \cup c_s$$

is an isomorphism.

Corollary 5.3. *When Γ is a free product of abelian groups, $H^*(\mathrm{W\small{H}}(\Gamma))$ is additively isomorphic to $H^*((\Gamma)^{n-1})$.*

6. The Case of Free Coxeter Groups

In this section we gain a deeper understanding of the ring $H^*(\mathrm{W\small{H}}(\Gamma))$ by continuing to narrow our focus. Here we compute the ring structure of $H^*(\mathrm{W\small{H}}(W_n); \mathfrak{K})$, where W_n, the n-fold free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$, is the 'free Coxeter group' and $\mathfrak{K}$ is a field of characteristic 2 (or simply the field with two elements). The results in the last section supply us with an additive answer. The ring structure comes from an adaptation of a result from Brownstein and Lee's paper [3].

Identify the canonical generators of W_n as $x_1, x_2, \ldots, x_n$. Elementary Whitehead automorphisms for $\mathrm{W\small{H}}(W_n)$ can be written as α_{ij}, where

$$\alpha_{ij}(x_k) = \begin{cases} x_k & \text{if } k \neq i \\ x_k^{x_j} & \text{if } k = i. \end{cases}$$

We note that the automorphisms α_{ij} are of order two. We denote classes in cohomology dual to these elements by α_{ij}^*.

Theorem 6.1. *(cf. Theorem 2.10 in* [3]*) Let $\mathfrak{K}$ be a field of characteristic 2. The cohomology algebra $H^*(\mathrm{W\small{H}}(W_3), \mathfrak{K})$ is generated by six classes in degree one, α_{ij}^*, $1 \leq i, j \leq 3$ and $i \neq j$, subject to the relations*

1. $\alpha_{ij}^*\alpha_{ji}^* = 0$*;*
2. $(\alpha_{kj}^*)^n(\alpha_{ji}^*)^m = \Big((\alpha_{kj}^*)^n - (\alpha_{ij}^*)^n\Big)(\alpha_{ki}^*)^m$ *for all m, n.*

Proof. For a fixed index k, there is a epimorphism $\mathrm{W\small{H}}(W_3) \to \mathrm{W\small{H}}(W_2)$ given by sending the elements α_{ik} and α_{ki} to the identity. This is a split map, which implies that $H^*(\mathrm{W\small{H}}(W_2), \mathfrak{K})$ is a summand of $H^*(\mathrm{W\small{H}}(W_3), \mathfrak{K})$. Since $\mathrm{W\small{H}}(W_2)$ is isomorphic to W_2, its cohomology consists of two one-dimensional polynomial classes with no nontrivial products. Relation 1 follows.

To establish Relation 2, consider the normal subgroup $\mathrm{Inn}(W_3)$ of inner automorphisms, generated by $\alpha_{21}\alpha_{31}$, $\alpha_{12}\alpha_{32}$, and $\alpha_{13}\alpha_{23}$. There is a short exact sequence

$$1 \to \mathrm{Inn}(W_3) \longrightarrow \mathrm{W\small{H}}(W_3) \xrightarrow{\phi} \mathrm{OW\small{H}}(W_3) \to 1.$$

Making arbitrary choices, we consider $\mathrm{OW\small{H}}(W_3)$ to be generated by the images under ϕ of α_{21}, α_{12}, and α_{13}; we denote these elements in $\mathrm{OW\small{H}}(W_3)$ by $\overline{\alpha_{21}}$, $\overline{\alpha_{12}}$, and $\overline{\alpha_{13}}$. This implies that ϕ sends α_{31}, α_{32}, and α_{23} to $(\overline{\alpha_{21}})^{-1}$, $(\overline{\alpha_{12}})^{-1}$, and $(\overline{\alpha_{13}})^{-1}$ respectively. There is an obvious backmap $i : \mathrm{OW\small{H}}(W_3) \to \mathrm{W\small{H}}(W_3)$ which shows that the short exact sequence is split. In addition, both $\mathrm{Inn}(W_3)$ and $\mathrm{OW\small{H}}(W_3)$ are isomorphic to W_3. (See the discussion in Example 2.4.)

Consider the copy of W_2 in $\mathrm{OW\small{H}}(W_3)$ generated by $\overline{\alpha_{21}}$ and $\overline{\alpha_{12}}$. From the first paragraph of this proof we know that the cohomology of this subgroup is a summand of $H^*(\mathrm{W\small{H}}(W_3))$. let $Y = \langle\overline{\alpha_{12}}\rangle$ and $Z = \langle\overline{\alpha_{21}}\rangle$ respectively. Both Y and Z are isomorphic to $\mathbb{Z}_2$, so both $H^*(Y)$ and $H^*(Z)$ are generated by single one-dimensional polynomial generators. We denote the cohomology generators by y and z. Now $\phi \circ i$ is the identity on $\mathrm{OW\small{H}}(W_3)$, so $H^*(Y * Z)$ is again a summand in the cohomology of $\mathrm{W\small{H}}(W_3)$. Thus, ϕ^* is injective and

$$\phi^*(y^m) = (\alpha_{12}^*)^m - (\alpha_{32}^*)^m$$
$$\phi^*(z^n) = (\alpha_{21}^*)^n - (\alpha_{31}^*)^n.$$

In addition, as $y^m z^n = 0$ in $H^*(Y * Z)$,

$$((\alpha_{12}^*)^m - (\alpha_{32}^*)^m)\,((\alpha_{21}^*)^n - (\alpha_{31}^*)^n) = 0$$

also holds in $H^*(\mathrm{W\small{H}}(W_3))$. This product can be expanded, and after the relation $\alpha_{12}^*\alpha_{21}^* = 0$ is applied, Relation 2 results. An identical argument for all other pairs of classes yields the other relations. □

Expanding on the argument for Theorem 6.1, there are surjections $\mathrm{W\small{H}}(W_n) \to \mathrm{W\small{H}}(W_{n-1})$ that result from sending the generators α_{ni} and α_{in} to the identity, and these maps are split. Brownstein and Lee use these surjections to show that relations similar to those in Theorem 6.1 hold in $H^*(\mathrm{W\small{H}}(\mathbb{F}_n), \mathbb{Z})$ for arbitrary n, and the same argument works for $H^*(\mathrm{W\small{H}}(W_n), \mathfrak{K})$. It remains to show that there are no other relations.

Theorem 6.2. *The cohomology algebra of $H^*(\mathrm{W\small{H}}(W_n))$ is generated by $n(n-1)$ one dimensional classes, α_{ij}^*, $1 \le i, j \le n$ and $i \ne j$, subject to Relations 1 and 2 in Theorem 6.1.*

Proof. The argument is a variant of one found in [6]. We compare the cohomology groups in the algebra defined by the given generators and relations and show that this count coincides with the additive count from before. We start by characterizing the cohomology classes in $H^*(\mathrm{W\small{H}}(W_n))$.

Given any monomial in the cohomology generators α_{ij}^*, the monomial can be rewritten in such a way that it has no cycling of indices, that is, so it has no term of the form

$$(\alpha^*_{ij})^{n_1}(\alpha^*_{jk})^{n_2}\dots(\alpha^*_{st})^{n_{m-1}}(\alpha^*_{ti})^{n_m}.$$

To see why, take a monomial consisting of (powers of) m generators with cyclic indices. Using Relation 2 and an induction argument, the product of powers of the first $m-1$ generators in the monomial can be written as

$$\left(\pm\sum(\text{various monomials in the first } m-2 \text{ generators})\right)(\alpha^*_{it})^{\text{some power}}.$$

When this expression is multiplied by the final term, $(\alpha^*_{ti})^{n_m}$, the product is 0 by relation 1.

We can make one more reduction. Take any monomial in the cohomology generators α^*_{ij}. By expanding Relation 2 to

$$(\alpha^*_{kj})^n(\alpha^*_{ki})^m = (\alpha^*_{kj})^n(\alpha^*_{ji})^m + (\alpha^*_{ij})^n(\alpha^*_{ki})^m$$

we see that we can rewrite the monomial so the first index in the cohomology generators is not repeated. Another induction argument, using the fact that cyclic indices can be avoided, shows that any monomial can be written as a sum of terms such that in each term no first index is repeated.

We now characterize a monomial basis for the cohomology classes in degree n. Since we can rewrite classes so no two generators have the same first index, monomials consist of a product of up to n distinct generators subject to this condition. As all α^* have dimension 1, the sum of the generators' powers is n. Finally, no monomial has cyclic indices.

We establish an isomorphism between these generating monomials and certain forests. For a given monomial, start with labeled vertices $[n]$, and for each generator α_{kl} draw a directed edge from the vertex labeled l to the vertex labeled k. Since the first index only occurs at most once in any monomial, the in-degee of each vertex is at most one. Also, as no monomial contains cyclic indices the resulting graph has no cycles, although it is possible for some vertices to be incident to no edges. We note that each tree in the resulting forest has a unique 'source' vertex, which we view as its root. Hence the resulting graph is a planted forest, with edges on each tree directed away from the root. This process can be reversed.

As in the proof of the Main Theorem we will count certain cohomology classes in two different ways and show that the counts agree. In particular, we count monomials whose associated forest has a fixed degree sequence. Take a monomial that is a product of powers of $n-k$ generators, $x = \times(\alpha^*_{ij})^{p_i}$. We form a degree sequence associated to x, $d = \{d_1, d_2, \dots, d_n\}$, by noting that d_j is the number of generators in the monomial whose second index is j. The sum $\sum_{i=1}^n d_i = n-k$ is the number of distinct α^*'s in the monomial, and by Lemma 4.4, there are k components in the forest associated to x. Also by Theorem 4.2, there are

$$\binom{n-1}{k-1}\binom{n-k}{d_1, d_2, \dots, d_n}$$

forests with the degree sequence d. Let $p_1, p_2, \dots, p_{n-k}$ be the powers of the generators α^* in the monomial. For this monomial to be in the degree m cohomology, we must have $\sum_{i=1}^{n-k} p_i = m$. Let f_m to be the total number of solutions to $p_1 + p_2 + \dots + p_{n-k} = m$ where $p_j \geq 0$. Then the total number of cohomology classes associated to the degree sequence d is

$$\binom{n-1}{k-1}\binom{n-k}{d_1, d_2, \ldots, d_n} f_m.$$

We would like to compare this count to the number of analogous classes in $H^*((W_n)^{n-1})$. If these counts are equal, the cohomology groups are additively isomorphic and we have found all of the necessary ring relations. Consider W_n as the free product $G_1 * G_2 * \cdots * G_n$, where each G_j is isomorphic to $\mathbb{Z}_2$. To make the appropriate cohomology comparison, we note that as α_{ij} is the automorphism associated to conjugation by x_j, we can identify the subgroup $\langle \alpha_{ij} \rangle$ with G_j, the jth copy of $\mathbb{Z}_2$. Therefore we will count the number of monomials, $x = \times(\alpha^*_{ij})^{p_i}$, in dimension m such that d_j of the α^* come from $H^*(G_j)$. (If we wish to make this comparison more explicit, we can refer to the copies of G_j in the $(n-1)$-fold product of W_n as G_{ij}, for $i = 1, 2, \ldots, n-1$.) Given a degree sequence d with $\sum_{i=1}^n d_i = n - k$, the number of ways to pick d_j copies of G_j from $(W_n)^{n-1}$ for all j is given by the multinomial coefficient

$$\binom{n-1}{d_1, d_2, \ldots, d_n, k-1}.$$

Once the copies of G_j that supply the cohomology generators are chosen, we still need to determine appropriate values of the powers p_i of the generators so that the resulting monomial is m-dimensional. As in the paragraph above, there are f_m possibilities for each assignment of groups G_j. Therefore, the total number of cohomology classes in degree m associated to the degree sequence d is $\binom{n-1}{d_1, d_2, \ldots, d_n, k-1} f_m$. However,

$$\begin{aligned}\binom{n-1}{d_1, d_2, \ldots, d_n, k-1} &= \frac{(n-1)!}{d_1! d_2! \ldots d_n! (k-1)!} \\ &= \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(n-k)!}{d_1! d_2! \ldots d_n!} \\ &= \binom{n-1}{k-1}\binom{n-k}{d_1, d_2, \ldots, d_n}.\end{aligned}$$

The process can also be reversed, which implies an additive isomorphism between $H^m(\mathrm{WH}(W_n))$ and $H^m((W_n)^{n-1})$ for all m. This means, in turn, that there can be no additional ring relations beyond those in Relations 1 and 2. □

The additive cohomology isomorphism suggests an obvious question: does the isomorphism extend to the cup product structure? We have the following answer.

Theorem 6.3. *The cohomology rings $H^*(\mathrm{WH}(W_n))$ and $H^*((W_n)^{n-1})$ are not ring-isomorphic.*

Proof. We will proceed by contradiction, assuming that a suitable rewriting of the classes yields the isomorphism. As the surjections $\mathrm{WH}(W_n) \to \mathrm{WH}(W_{n-1})$ are split it is sufficient to show that the ring relations do not hold in $\mathrm{WH}(W_3)$.

Denote the canonical generators of $W_3 \times W_3$ by a_1, a_2, and a_3; and b_1, b_2, and b_3. Let a^*_i and b^*_i be the one-dimensional cohomology classes dual to a_i and b_i. By way of analyzing the cup product structure of $H^*(W_3 \times W_3)$, let x^* and y^* be arbitrary non-zero classes in $H^1(W_3 \times W_3)$. Distinguish classes from distinct free group factors by writing $x^* = A_1 + B_1$, and $y^* = A_2 + B_2$, where the A_j are sums of the a_i^* from the first free factor and the B_j are sums of the b_i^* from the second. We note that

$$x^* y^* = (A_1 + B_1)(A_2 + B_2) = A_1 B_2 + B_1 A_2.$$

If this product is zero, then $A_1 B_2 + B_1 A_2 = 0$, so

$$A_1 B_2 = -B_1 A_2 = A_2 B_1.$$

When $x^* \neq y^*$, this only happens if either A_1 and A_2 are both zero (so $x^* = B_1$ and $y^* = B_2$), or if both B_1 and B_2 are both zero (so $x^* = A_1$ and $y^* = A_2$). That is, if x^* and y^* are distinct one-dimensional classes that cup to zero, then they must be classes associated to the same copy of W_3 in $W_3 \times W_3$.

Recall that $H^*(\mathrm{W\small{H}}(W_3))$ is generated by six one-dimensional classes, α_{ij}^*. From Relation 1 in Theorem 6.1, we know that $\alpha_{ij}^* \alpha_{ji}^* = 0$, Thus α_{21}^* and α_{12}^* must denote classes from the same copy of W_3, and similarly with α_{13}^* and α_{31}^* as well as α_{32}^* and α_{23}^*. At least two of these pairs of classes must be associated with the same free group. Without loss of generality, let α_{12}^*, α_{21}^*, α_{13}^* and α_{31}^* be these classes. Then $\alpha_{21}^* \alpha_{31}^* = 0$. However, $\alpha_{21}^* \alpha_{31}^*$ is a non-zero class in $H^2(\mathrm{W\small{H}}(W_3))$ which is impossible. □

Remark 3. The results in this section should generalize to the case where Γ is a free product of arbitrary (finitely generated) abelian groups.

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